Long-Run Comparative Statics

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Abstract

How do permanent changes in productivity, policies, and rates of return affect long-run consumption after capital has fully adjusted? We show that the balanced growth paths of a broad class of dynamic, multisector, open economies can be represented as equilibria of equivalent distorted static economies. In these equivalent economies, capital services are produced instantaneously from investment goods and sold at an as-if markup equal to the ratio of capital income to investment expenditure. This markup measures the economy's deviation from the Golden Rule of savings. We use this representation to characterize long-run comparative statics in terms of expenditure shares, substitution elasticities, and initial wedges. We decompose the consumption effect of an industry-level productivity change into two components: a technological effect, measured by a cost-based Domar weight that incorporates both production and investment networks, and a reallocation effect, summarized by induced changes in aggregate labor-income shares. We show that the long-run consumption effects of distortions satisfy a dynamic analogue of Harberger's formula: induced changes in capital stocks are weighted by the gap between capital income and investment expenditure. We also develop a small open economy analogue that can be used without requiring the solution of the full world equilibrium.

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1 Introduction

Many macroeconomic questions concern the long-run effects of permanent changes to the economy. How much do permanent changes in productivity and tax policy affect aggregate consumption in the long run, taking into account the adjustment of the capital stock? And which microeconomic statistics are sufficient to quantify these effects? Such questions are central to empirical macroeconomics. In cross-sectional work, researchers investigate how permanent differences in policies or technologies explain persistent differences in outcomes between countries or regions. In time-series analysis, they ask how changes in technologies or economic regimes shift the long-run level of consumption or GDP.[1] Although these long-run outcomes are not direct measures of date-0 welfare, they remain central empirical objects for a practical reason: they are much easier to measure. Evaluating changes in dynamic welfare requires taking a stand on things like discount factors and social welfare weights, whereas long-run shifts in consumption or GDP can be quantified using standard national accounts data.[2]

Despite their empirical importance, the literature lacks a general characterization of long-run outcomes in dynamic general equilibrium models. This contrasts sharply with the analysis of static networked economies, where a rich literature provides powerful characterization results that map microeconomic parameters to aggregate outcomes in general settings (Harberger, 1964; Hulten, 1978; Baqaee and Farhi, 2019; Liu, 2019; Bigio and La'O, 2020; Baqaee and Farhi, 2020). This paper develops an analogue of these results for long-run outcomes in dynamic economies.

Our starting point is an equivalence between balanced growth paths in dynamic economies and equilibria in static economies. In Section 2, we use the neoclassical growth model to illustrate the basic idea: prices and quantities along a balanced growth path (BGP) can

[1] For example, the development accounting literature seeks to explain cross-country differences in outcomes – such as hours worked and GDP per capita – by appealing to differences in taxes, endowments, and technologies (e.g., Prescott, 2004; Caselli, 2005; Hsieh and Klenow, 2010; Lagakos and Schoellman, 2026). Increasingly, development accounting exercises emphasize the importance of input-output networks and complementarities (Jones, 2011; Bartelme and Gorodnichenko, 2015; Caliendo et al., 2022; Colmer et al., 2024). In the time-series, quasi-experimental methods like difference-in-differences and synthetic controls are used to estimate the long-run impact of permanent shocks, such as joining the EU (Grassi, 2024) or Brexit (Born et al., 2019), on consumption, investment, and GDP.

[2] Aggregate consumption and GDP can be defined in terms of contemporaneous prices and quantities, and are routinely constructed by statistical agencies. In contrast, intertemporal welfare is notoriously difficult to measure empirically. Its calculation requires researchers to make assumptions about numerous unobservables, including the expected path of future prices and quantities, social welfare weights for different households or generations, and the specific rates used to discount the future. Recent work continues to explore these complexities from both a macro perspective (e.g., Basu et al., 2022, assuming a representative agent) and a micro perspective that allows for household heterogeneity (Del Canto et al., 2023; Fagereng et al., 2022; Baqaee et al., 2024).

be reproduced as the equilibrium of an as-if static economy with wedges. In this representation, capital services are produced instantaneously from investment goods and sold at an as-if markup equal to the ratio of capital income to investment expenditure. Since capital income is the revenue generated by the capital stock and investment is the cost of maintaining it, the ratio between them is isomorphic to a markup, with the size of the wedge quantifying the deviation from the Golden Rule of savings.[3]

In Section 3, we set up a general multi-country, multi-industry model and define balanced-growth-path equilibria. In Section 4, we show that the equivalence between BGPs and distorted static equilibria is preserved in this more general setup. This means that the long-run effect of a permanent change to any parameter can be characterized by a corresponding shift in the parameters and wedges of the as-if static model. This allows us to import the analytical toolkit developed for distorted static economies to characterize balanced growth comparative statics. Long-run comparative statics depend on the same objects that appear in distorted static economies: initial expenditure shares, wedges, and elasticities of substitution.

In Section 5, we characterize changes in global consumption, defined as the expenditure-weighted change in consumption across countries. Global consumption is particularly tractable because terms-of-trade gains and losses cancel across countries; in a closed economy, global and domestic consumption coincide.

We show that the long-run consumption response to a productivity improvement can be decomposed into two components. The first is a technological effect: the increase in consumption that would occur if industries continued to allocate their output across downstream uses in the same proportions as before. The second is a reallocation effect: the additional change in consumption generated by the shifts in input use and production patterns induced by changes in relative prices. This decomposition separates the mechanical propagation of a productivity shock through the production and investment networks from the reallocation due to general equilibrium effects.

The technological effect is equal to an easy-to-compute cost-based Domar weight. This weight measures an industry's contribution to the cost of producing consumption, both directly and indirectly through the production network and the network of investment flows. It can be constructed from input-output and investment-flow data independent of elasticities of substitution and the household savings block. When production and preferences are Cobb-Douglas, there is no reallocation effect, and the cost-based Domar weight gives the full long-run consumption elasticity. Unlike a conventional sales-based Do-

[3] This is the same criterion as the one introduced by Abel et al. (1989) for determining whether the capital stock is below its Golden Rule value.

mar weight, it places greater weight on investment-good producers and their upstream suppliers. The reason is that, away from the Golden Rule, capital income exceeds the investment expenditure required to maintain the capital stock, so observed sales understate the importance of industries that contribute to capital formation.

When productivity shocks induce reallocation, that effect is measured by the reduction in some average of aggregate labor income shares. Intuitively, a decline in the labor share indicates that resources have shifted toward more capital-intensive or more highly-wedged uses. When the economy's capital stock is below the Golden Rule, such reallocations raise long-run consumption because they expand activities whose capital income exceeds their investment cost. The sign of this effect depends on substitution patterns. For example, investment-specific technical change generates a positive reallocation effect when capital and labor are gross substitutes, but a negative one when they are gross complements. Cobb-Douglas production is the neutral case in which labor shares remain constant and the reallocation term vanishes.

Section 5 also shows how long-run world consumption responds to distortions and capital taxes. Changes in markups, tariffs, and capital taxes all operate as changes in wedges in the equivalent static economy. Their effects on long-run consumption therefore take the form of a dynamic analogue of Harberger's formula (Harberger, 1964): consumption rises when a policy change expands activities with high initial wedges and falls when it contracts them. The terms that are distinctive for dynamic economies arise from capital accumulation. Changes in capital stocks are weighted by the distance to the Golden Rule, measured by the gap between the rental income and investment expenditures. This means that in undistorted perfectly competitive economies, policy changes have first-order effects on long-run consumption only if capital income is not equal to investment expenditures.

Tariffs provide a particularly transparent application. Starting from free trade, a symmetric increase in tariffs raises the relative price of investment goods and depresses capital accumulation. When capital income exceeds investment expenditure — that is, when capital is below its Golden Rule level — the resulting decline in capital stocks lowers long-run consumption to first order. This mechanism differs from the channels emphasized in static trade models. Its magnitude is governed primarily by the response of capital demand, especially the elasticity of substitution between capital and labor, rather than by the trade elasticity that determines substitution across countries.[4]

[4]When we extend the model in Section 7 to allow for endogenous returns, there is also a role for the elasticity of capital supply, defined as the savings response to rates of return, since this elasticity shapes the response of the capital stock.

The Golden Rule provides a useful benchmark that unifies these results. When each capital stock is at its Golden Rule level and there are no other distortions, capital income is equal to investment expenditures, as-if capital wedges disappear, and the equivalent static economy is undistorted. Cost-based and sales-based Domar weights then coincide, and first-order reallocation effects disappear due to the envelope theorem. For productivity shocks, we recover a dynamic version of Hulten's theorem: the long-run consumption effect of a sectoral productivity improvement is given by the sector's sales relative to consumption, independently of substitution elasticities and the structure of the production network beyond what is summarized by sales. Likewise, small changes in distortions or required rates of return have no first-order effect on long-run consumption.

Section 6 focuses on country-level consumption in open economies. In general, a productivity shock in one country changes foreign prices, foreign quantities, and export demand, which then feed back into domestic consumption through trade. This means that the response of consumption in each country will depend on the full structure of the entire world economy. To obtain a more tractable and less information-intensive benchmark, we use a small open economy approximation in which each country takes import prices and isoelastic export-demand schedules as given. We show that this economy can itself be represented as an equivalent distorted static closed economy. In this representation, imports are produced linearly from export revenue, while export revenue is "produced" from domestic exports using a decreasing-returns technology and a wedge capturing the terms-of-trade externality. This representation yields a small open economy cost-based Domar weight. Under a Cobb-Douglas benchmark, this weight gives the effect of domestic productivity changes on domestic consumption. These weights can be constructed from domestic input-output and investment-flow data, together with export-demand elasticities, without specifying or solving the full world equilibrium. In our quantitative model of the world economy, these simple cost weights closely approximate the full-model response of long-run domestic consumption to permanent changes in sectoral productivity.

In Section 7, we relax the assumption of closed international capital markets and infinitely-lived representative agents in each country. In the extended model, trade imbalances, interest rates and risk premia adjust endogenously, rather than being pinned down as simple functions of exogenous parameters. The static representation of BGPs continues to apply: conditional on rates of return and net foreign asset positions, the balanced growth path is represented by the same equivalent static economy, with returns determining the as-if capital wedges and net foreign asset income appearing as transfers across countries. In addition, there are auxiliary asset-market-clearing conditions that determine rates of return and net foreign asset positions. The additional information required for comparative statics consists of initial net foreign asset positions and the elasticities of household asset demand with respect to returns.

In Section 8, we assess the quantitative relevance of these results in a calibrated model of the world economy. The model combines a standard multi-country, multi-sector trade structure (Costinot and Rodriguez-Clare, 2014) with international input-output linkages, industry-specific investment flows, and capital accumulation. We calibrate it using expenditure shares from the World Input-Output Database (Timmer et al., 2015), augmented with the global investment-flow tables of Ding (2022). We first study permanent sectoral productivity changes. For global consumption, the cost-based Domar weight closely approximates the response obtained from the full model. For country-level consumption, its small open economy analogue performs similarly well, despite requiring only domestic input-output and investment-flow data together with export-demand elasticities. Both measures substantially outperform the conventional approach of scaling TFP by the inverse of the labor share, particularly for investment-producing industries and their upstream suppliers.

We then study the long-run effects of tariffs. In the benchmark calibration, a uniform 20% tariff lowers long-run global consumption by about 3 percent. The main mechanism is not the static reallocation of expenditure across countries, but the increase in investment prices and resulting decline in capital accumulation. In line with our analytical results, the first-order loss vanishes when capital is held fixed or when the economy begins at the Golden Rule. Its magnitude depends primarily on the elasticity of substitution between capital and labor and, when returns are endogenous, on the elasticity of household asset demand. By contrast, varying the trade elasticity has little effect. At the country level, adjustment in capital stocks likewise accounts for most of the consumption response, while terms-of-trade effects are quantitatively small.

Related Literature. This paper contributes to the literature on sufficient-statistic characterizations of aggregate outcomes in economies with production networks. Harberger (1964), Hulten (1978), and more recent work including Baqaee and Farhi (2019, 2020, 2024), Bigio and La'O (2020), Liu (2019), and Dávila and Schaab (2023) characterize the effects of productivity and distortions in static economies using expenditure shares, wedges, and substitution elasticities.[5] We extend this approach to long-run outcomes in dynamic economies with capital accumulation. Our equivalence result shows that balanced-growth comparative statics can be analyzed using the same tools once capital accumulation is

[5] More broadly, the paper relates to the literature on the propagation of sectoral shocks through production networks, including Foerster et al. (2011), Gabaix (2011), Acemoglu et al. (2012), Di Giovanni et al. (2014), and Buera and Trachter (2024).

represented by as-if capital wedges. This yields dynamic analogues of Hulten's productivity formula and Harberger's misallocation formula and clarifies how production and investment networks jointly determine long-run consumption responses.

We also contribute to the literature on dynamic multisector and international general equilibrium models. Foerster et al. (2022) work with a closed-economy Cobb-Douglas model with an infinitely-lived representative agent. Our analysis relaxes these assumptions by allowing for an open economy, an arbitrary elasticity structure, imperfectly elastic capital supply, and heterogeneous returns across sectors. Ding (2022) constructs investment flow tables for the world economy, and uses this dataset to study the gains from trade relative to autarky allowing for adjustments in capital. We relax the assumptions of infinitely-lived agents, zero growth, financial autarky, and homogeneous returns across capital goods. We also characterize the response of the economy to a different set of counterfactuals.

Our paper is also related to the broader literature on dynamic disaggregated and international general equilibrium models, pioneered by Long and Plosser (1983) and Backus et al. (1992). Some recent contributions include Alvarez (2017), Kehoe et al. (2018), Ravikumar et al. (2019), Dix-Carneiro et al. (2023), Lyon and Waugh (2019), Vom Lehn and Winberry (2022), and Kleinman et al. (2023).[6] We complement this literature by providing analytical characterizations for balanced-growth outcomes.

Our paper is related to a long tradition in the treatment of investment and capital in national income accounting. One recurring theme in this literature is that since consumption is the only true final good, investment goods should be viewed as intertemporal intermediate inputs (Kuznets, 1941; Hulten, 1979), prompting suggestions for different ways of netting out investment costs from GDP to be consistent with the treatment of other intermediates (Weitzman, 1976; Barro, 2021). Our paper provides a precise sense in which capital goods are equivalent to intermediates for the purpose of long-run comparative statics.

Finally, our extension with endogenous returns and international asset trade relates to the intertemporal approach to the current account (Obstfeld and Rogoff, 1995) and to models in which investment risk and financial development shape global imbalances (Caballero et al., 2008; Mendoza et al., 2009; Angeletos and Panousi, 2011; Cuñat and

[6]In this paper, we abstract from fixed costs and entry/exit decisions of firms, for example, as in Hopenhayn (1992) and Melitz (2003). Alessandria et al. (2021) review this literature as it pertains to international trade. Boehm et al. (2024) show how to extend formulas in the style of Arkolakis et al. (2012) to dynamic models with firm entry. Barkai and Panageas (2021) study how the distribution of the types of entering firms affects long-run consumption near the Golden Rule. Although we do not explicitly study this class of models, we provide an example of how our results can be extended to models with firm entry in the appendix.

Zymek, 2024). We allow long-run asset demand to be imperfectly elastic, so that rates of return, risk premia, and net foreign asset positions respond endogenously to shocks. We show that these forces can be incorporated without abandoning the static representation: conditional on returns and foreign asset positions, the balanced-growth allocation is still the equilibrium of an equivalent distorted static economy, while asset-market-clearing conditions determine the associated capital wedges and cross-country transfers. Comparative statics therefore require only the baseline sufficient statistics, supplemented by asset-demand elasticities and initial net foreign asset positions, and do not require solving the transition path.

2 Illustration in the Neoclassical Growth Model

A key idea in our paper is that balanced-growth paths are equivalent to equilibria of static economies with wedges. Here, we illustrate this result in the simplest possible setting: the standard one-sector neoclassical growth model. Later, we show that this equivalence extends to a much larger class of models, and use it to characterize long-run comparative statics.

Consider the following equations:

Balanced growth path	**Equivalent static economy**
$Y = F(K, L) = C + X,$	$Y = F(K, L) = C + X,$
$X = (g + \delta)K,$	$K = A_K X,$
$w = F_L(K, L),$	$w = F_L(K, L),$
$r + \delta = F_K(K, L),$	$\mu_K / A_K = F_K(K, L),$
$C = wL + (r - g)K.$	$C = wL + (\mu_K - 1)X.$

The left-hand side is a standard characterization of balanced growth paths in the neoclassical growth model in continuous time, with all quantities being normalized by the common growth trend e^{gt}. Here, Y is output, K and L are capital and labor, C is consumption, X is investment, g is the growth rate, and δ is the depreciation rate. The first equation is the resource constraint, stating that total output equals consumption plus investment. The second is the relationship between investment and the capital stock on the balanced growth path. The third and fourth equations are firms' first-order conditions. The last equation follows from the household's budget constraint.

The equations on the right-hand side are equilibrium equations for a static economy.

The first equation states that gross output Y is used as consumption C or as an intermediate input X. The second equation states that this intermediate X is converted into a different good K using a linear technology with TFP shifter A_K. The third and fourth equations are the first-order conditions for the firm, assuming that K is sold at a markup μ_K over its marginal cost of production $1/A_K$. The final equation is the household's budget constraint stating that consumption is financed by labor income and revenues generated by the markup on X.

These equations coincide if we let $A_K = 1/(g+\delta)$ and $\mu_K = (r+\delta)/(g+\delta)$, leading to the following observation.

Proposition 1 (Equivalent Static Economy)**.** *The balanced growth path prices and quantities in the neoclassical growth model coincide with the equilibrium prices and quantities of a static economy where capital is produced instantaneously from investment goods with productivity* $A_K = 1/(g+\delta)$ *and sold at a markup* $\mu_K = (r+\delta)/(g+\delta)$.

Capital compensation $(r+\delta)K$ and investment $(g+\delta)K$ on the balanced growth path map to the revenues and costs of the K industry in the static economy. Hence, their ratio maps to the markup μ_K and their difference, which is net capital income $(r-g)K$, maps to profit income $(\mu_K - 1)X$.

The capital wedge is present in the static economy since the dynamic economy does not maximize the level of consumption on the balanced growth path. In order for the static economy to replicate the balanced growth path allocation, there needs to be a wedge on capital to explain why consumption is not maximized. Since the wedge on capital reflects discounting, its presence in the static economy does not imply that the dynamic equilibrium is inefficient. At the Golden Rule of savings, which is the level of savings that maximizes long-run consumption, $r = g$, gross capital income equals investment, and the capital wedge in the static economy disappears.

For the one-good neoclassical model, the static representation is largely a relabeling. However, the equivalence between balanced-growth paths and distorted static economies is very general. In the next two sections, we establish a similar equivalence in a model with multiple industries, input-output linkages, multiple capital goods, and international trade. This equivalence then makes it possible to apply tools developed for static distorted economies to study balanced-growth paths of dynamic economies.

3 Model

This section sets up our baseline model and its balanced growth equations. The framework is designed to speak to the literatures on trade, production networks, and development accounting. The baseline model features infinitely-lived agents and closed international capital markets; Section 7 presents an extension with richer models of asset accumulation. Other extensions are treated in Appendix B.2 (non-homothetic preferences, firm entry, biased productivity growth, endogenous growth rates, and labor-leisure choice).

3.1 Environment

Consider a continuous-time, dynamic multi-country model with capital accumulation. There is a set of countries $h \in \mathcal{H}$, with each country populated by an infinitely-lived representative household that owns the domestic capital stock. We assume that there is no international borrowing or lending: investment is financed by local savings (Section 7 relaxes this assumption). Each country h produces a final consumption bundle C_h, intermediate inputs Y_{hi}, and investment goods X_{hk}. Each investment good is used to accumulate an associated capital stock K_{hk}. Each country also has a set of labor endowments, with L_{hf} denoting the efficiency units of labor type f in country h. All labor endowments grow at a common, constant rate g, with g capturing both population growth and labor-augmenting technical change. We study balanced growth paths along which quantities grow at rate g while prices and rates of return are constant.

3.2 Balanced-growth equations

Below we present the balanced-growth equations that are the focus of the analysis. For the household block, we show how these equations follow from a dynamic household problem; for the remaining blocks, we state them directly. Appendix A.1 presents the full time-dependent economy.

Households. Each country is populated by a representative household with preferences

$$U_h = \int_0^\infty e^{-\rho t} \frac{C_h(t)^{1-1/\gamma} - 1}{1 - 1/\gamma} \, dt, \tag{1}$$

where γ is the intertemporal elasticity of substitution and ρ is the discount rate. The consumption good $C_h(t)$ is a constant-returns aggregate of intermediate goods with a price index $p^C_h(t)$. The household budget constraint is

$$p^C_h(t)C_h(t) + \sum_{k\in\mathcal{K}} \dot{B}_{hk}(t) \leq \sum_{f\in\mathcal{F}} w_{hf}(t)L_{hf}e^{gt} + \sum_{k\in\mathcal{K}} (r_{hk}(t) - \tau^K_{hk})B_{hk}(t) + T_h(t),$$

where there is one asset $B_{hk}(t)$ for each capital good, with households only being able to invest in domestic capital goods. Each capital good has a return $r_{hk}(t)$ and an associated wealth tax τ^K_{hk}, which is a reduced-form way to allow for heterogeneous returns across capital goods.[7] The labor income term encodes the assumption that all labor endowments grow at a common rate g, and $T_h(t)$ denotes any other income.

On a balanced growth path, prices are constant, and so are quantities normalized by e^{gt}, implying a time-invariant budget constraint:

$$p^C_h C_h = \sum_{f\in\mathcal{F}} w_{hf}L_{hf} + \sum_{k\in\mathcal{K}} (r_{hk} - \tau^K_{hk} - g)B_{hk} + T_h. \tag{2}$$

The household Euler equation and the absence of arbitrage imply that rates of return on a balanced growth path satisfy $r_{hk} = \tau^K_{hk} + \rho + g/\gamma$.

Production. All goods are produced with constant-returns-to-scale production functions. Dropping the time index, quantities on the balanced growth path satisfy:

$$Y^C_h = A^C_h F^C_h \left[\{Y^C_{h,h'j}\}_{h'\in\mathcal{H},j\in\mathcal{N}} \right], \qquad h \in \mathcal{H}, \tag{3}$$

$$Y^Y_{hi} = A^Y_{hi} F^Y_{hi} \left[\{Y^Y_{hi,h'j}\}_{h'\in\mathcal{H},j\in\mathcal{N}}, \{K_{hi,hk}\}_{k\in\mathcal{K}}, \{L_{hi,hf}\}_{f\in\mathcal{F}} \right], \qquad h \in \mathcal{H},\ i \in \mathcal{N}, \tag{4}$$

$$X_{hk} = A^X_{hk} F^X_{hk} \left[\{Y^X_{hk,h'i}\}_{h'\in\mathcal{H},i\in\mathcal{N}} \right], \qquad h \in \mathcal{H},\ k \in \mathcal{K}. \tag{5}$$

Equation (3) is the production function for the consumption bundle, which only uses intermediate goods as inputs, potentially from all countries (iceberg trade costs are captured as productivity penalties inside the production function). Equation (4) is the production function of intermediate inputs, which uses other intermediates from potentially all countries, as well as local capital services and labor inputs. Equation (5) is the production function of investment goods, which uses only local and foreign intermediate inputs. We order subscripts buyer first, seller second: for example, $Y^Y_{hi,h'j}$ is the intermediate input that industry i in country h purchases from industry j in country h'. The terms A^C_h, A^Y_{hi}, A^X_{hk} are productivity shifters.

[7] In Section 7, we study a setting where heterogeneous returns can be microfounded as risk premia.

Capital accumulation. Each investment good X_{hk} has an associated capital good K_{hk}, which is accumulated linearly from X_{hk} with depreciation rate δ_k . On a balanced growth path, the relationship between capital stock and the flow of investment is

$$X_{hk} = (g + \delta_k)K_{hk}, \quad h \in \mathcal{H},\ k \in \mathcal{K}, \tag{6}$$

since sustaining one unit of capital on the BGP requires a flow of $g + \delta_k$ units of the investment good to offset depreciation and keep up with the growth rate.

Prices. Firms minimize costs taking intermediate input prices, capital rental rates, and wages as given. Given the resulting unit costs, prices are given by

$$p^C_h = \mathcal{C}^C_h, \qquad p^Y_{hi} = \mu^Y_{hi}\,\mathcal{C}^Y_{hi}, \qquad p^X_{hk} = \mathcal{C}^X_{hk}, \tag{7}$$

where the $\mathcal{C}$'s denote the unit cost functions of the different producers. The term μ^Y_{hi} is a wedge taking the form of a markup between marginal costs and prices of intermediate inputs, and is a reduced-form way to capture distortions such as tariffs, monopoly power, and other forms of resource misallocation.[8] We adopt the convention that only intermediate inputs have markups, with distortions on consumption or investment goods captured by markups on the underlying intermediate inputs.[9]

User cost of capital. The rental rate of capital is given by

$$R_{hk} = (r_{hk} + \delta_k)\, p^X_{hk}, \tag{8}$$

and follows from a standard asset market arbitrage argument (see the discussion of asset market clearing below).

Wedge and transfer income. Wedge income from the markups μ^Y_{hi} and the wealth tax τ^K_{hk} is rebated to household h. To allow for unbalanced trade on a balanced growth path, we also allow for an international transfer $\mathcal{T}_h$ in units of the numeraire, with $\sum_h \mathcal{T}_h = 0$. The

[8]We treat μ^Y as exogenous. In general, wedges may be endogenous — for example, if there are variable markups under Kimball demand or quantity restrictions. The comparative statics we derive below then apply alongside auxiliary equations pinning down changes in wedges; see Baqaee and Farhi (2020) for details.

[9]This is mainly a matter of convenient representation. Effective wedges on consumption or investment goods can be represented by placing a tax on a middleman that buys intermediate inputs and supplies them to the consumption or investment industries. In Section 8, we use such a setup to model bilateral tariffs.

resulting income from wedge revenues and international transfers is given by:

$$T_h = \sum_{k \in \mathcal{K}} \tau^K_{hk} B_{hk} + \sum_{i \in \mathcal{N}} (1 - 1/\mu^Y_{hi})\, p^Y_{hi} Y^Y_{hi} + \mathcal{T}_h, \qquad h \in \mathcal{H}. \tag{9}$$

Resource constraints. Resource constraints for factors and goods are given by

$$Y^C_h = C_h, \qquad h \in \mathcal{H}, \tag{10}$$

$$Y^Y_{hi} = \sum_{h' \in \mathcal{H}} \left[Y^C_{h',hi} + \sum_{i' \in \mathcal{N}} Y^Y_{h'i',hi} + \sum_{k \in \mathcal{K}} Y^X_{h'k,hi} \right], \qquad h \in \mathcal{H}, i \in \mathcal{N}, \tag{11}$$

$$X_{hk} = K_{hk}(g + \delta_k), \qquad h \in \mathcal{H}, k \in \mathcal{K}, \tag{12}$$

$$K_{hk} = \sum_{i \in \mathcal{N}} K_{hi,hk}, \qquad h \in \mathcal{H}, k \in \mathcal{K}, \tag{13}$$

$$L_{hf} = \sum_{i \in \mathcal{N}} L_{hi,hf}, \qquad h \in \mathcal{H}, f \in \mathcal{F}, \tag{14}$$

where the left-hand side is the supply of goods and factors, and the right-hand side is their use as final consumption, inputs to production, and investment.

Asset market clearing. Each country has a perfectly competitive financial intermediary that collects household savings B_{hk}, invests them in capital goods, and rents out capital services to firms; see Appendix A.1 for details. On a balanced growth path, the user cost equation (8) follows from a standard no-arbitrage condition on the intermediary's portfolio choices. Furthermore, asset market clearing and the household Euler equation yield:

$$B_{hk} = p^X_{hk} K_{hk}, \qquad h \in \mathcal{H}, k \in \mathcal{K}, \tag{15}$$

$$r_{hk} = \rho + g/\gamma + \tau^K_{hk}, \qquad h \in \mathcal{H}, k \in \mathcal{K}. \tag{16}$$

Given the equations above, we introduce the following definition.

Definition 1 (Balanced Growth Path)**.** A balanced growth path is a set of quantities and prices that satisfy equations (2)–(16).

4 Equivalence Result and Comparative Statics

We are interested in how changes in productivities A, wedges μ^Y, and wealth taxes τ^K affect balanced growth outcomes, starting from an initial balanced growth path with (A_0, μ^Y_0, τ^K_0). We assume that balanced growth paths are locally smooth, unique, and attractive around (A_0, μ_0^Y, τ_0^K), so that parameter changes shift the economy to a new balanced growth path, as illustrated in Figure 1. We also assume that the exogenous transfers $\mathcal{T}_h$ are fixed as a share of global consumption spending (Section 7 endogenizes the international transfers).

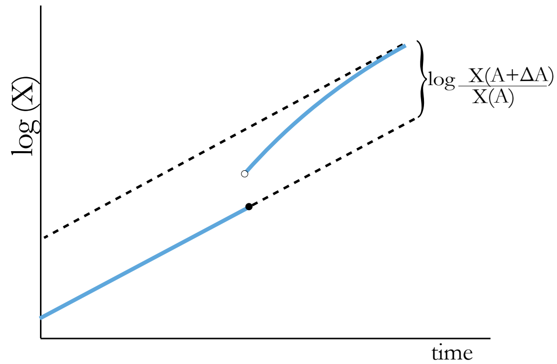

Figure 1: Long-run comparative static: a permanent parameter change shifts the economy from one balanced growth path to another.

Let $\mathcal{X}^{BGP}(A, \mu^Y, \tau^K)$ be a function that maps productivities, wedges, and return shifters into the vector of balanced-growth-path quantities and prices.[10] Define *long-run comparative statics* as

$$\mathcal{X}^{BGP}(A, \mu^Y, \tau^K) - \mathcal{X}^{BGP}(A_0, \mu_0^Y, \tau_0^K).$$

We characterize these comparative statics in two steps. Section 4.1 establishes that balanced growth path allocations coincide with equilibria of a distorted static economy, reducing the dynamic problem to a static one. The equivalence means that comparative statics can be expressed in terms of a few sufficient statistics: expenditure shares, elasticities of substitution, and initial wedges. Section 4.2 presents these objects, and Section 4.3 uses them to characterize long-run comparative statics.

4.1 Equivalence Result

The following equivalence result establishes the existence of an equivalent static economy such that balanced growth allocations are equilibria of that economy.

Proposition 2 (Equivalence between BGPs and distorted static economies)**.** *Given parameters A, μ^Y, and $\boldsymbol{\tau}^K$, the balanced growth path prices and quantities $\mathcal{X}^{BGP}(A, \mu^Y, \boldsymbol{\tau}^K)$ are the same as the prices and quantities of a static economy. In this static economy, (i) the wedges μ^Y*

[10]Given the smoothness of the underlying system, $\mathcal{X}^{BGP}$ is a well-defined smooth function in a neighborhood of (A_0, μ_0^Y, τ_0^K) as long as the balanced growth path is locally unique. Given our assumption of local attractiveness, the economy converges to $\mathcal{X}^{BGP}(A, \mu^Y, \tau^K)$ starting from $\mathcal{X}^{BGP}(A_0, \mu_0^Y, \tau_0^K)$.

and the production functions for consumption, intermediate, and investment goods are the same as in the dynamic economy; (ii) capital stocks K_{hk} are replaced by capital services that are produced instantaneously from investment goods according to the production function $K_{hk} = A^K_{hk} X_{hk}$, with productivity shifter $A^K_{hk} = 1/(g + \delta_k)$; (iii) capital services are sold at a markup $\mu^K_{hk} = (r_{hk} + \delta_k)/(g + \delta_k)$, with markup revenue from capital good k in country h distributed to households in that country and $r_{hk} = \rho + g/\gamma + \tau^K_{hk}$.

Proposition 2 allows us to study long-run comparative statics by studying comparative statics in the equivalent static economy. The logic is similar to the one in the neoclassical growth model in Section 2. The balanced growth accumulation equation (6) is identical to a linear production function that converts investment goods into capital services with productivity $1/(g + \delta_k)$. This technology implies that the marginal cost of producing capital services is $p^X_{hk}(g + \delta_k)$. Further, the rental price of capital is $p^X_{hk}(r_{hk} + \delta_k)$ by Equation (8); the ratio between this rental price and the marginal cost of producing capital services maps to the markup μ^K_{hk} in the proposition.

Just as in the neoclassical growth model in Section 2, the capital wedges μ^K_{hk} capture the effects of discounting on investment and their presence does not imply that the dynamic economy is inefficient. The wedges collapse to one at the Golden Rule, when $r_{hk} = g$ for all capital goods. The wedges also collapse to one when $\delta_k \to \infty$, since capital is then no longer durable, and therefore behaves like a regular intermediate input (because the model is formulated in continuous time, the limit $\delta \to \infty$ corresponds to full depreciation). Empirically, μ^K_{hk} is the ratio of capital income to investment on the balanced growth path, and it exceeds unity whenever capital income exceeds investment.[11]

4.2 Sufficient statistics

Comparative statics in distorted static economies can be represented in terms of a few sufficient statistics: expenditure shares, price elasticities, and initial wedges (Jones, 1965; Baqaee and Farhi, 2024). Since Proposition 2 shows balanced growth paths are equivalent to such economies, long-run comparative statics can be characterized using the same type of sufficient statistics. We introduce these objects here; Section 4.3 uses these objects to provide a full characterization of long-run comparative statics.

As in Section 3, we partition the economy into four types of sectors:

- **Consumption goods:** Final consumption industries, indexed by countries $h \in \mathcal{H}$.

[11] This is the same criterion for capital being below its Golden Rule level as in Abel et al. (1989).

- **Intermediate inputs:** Intermediate input producers, indexed by country $h \in \mathcal{H}$ and industry $i \in \mathcal{N}$.
- **Investment and capital goods:** Industries that produce investment goods, and their associated capital stocks, with each country h having an investment good and its associated capital stock, indexed by hk, for each $k \in \mathcal{K}$.
- **Labor:** Labor inputs, indexed by country $h \in \mathcal{H}$ and labor type $f \in \mathcal{F}$.

Using this partition, we construct four objects: expenditure shares, a matrix of input cost shares, an array of input elasticities of substitution, and a vector of initial wedges.

1. Expenditure shares. The country share Φ_h of world consumption is defined as

$$\Phi_h = \frac{p_h^C C_h}{\sum_{h' \in \mathcal{H}} p_{h'}^C C_{h'}} \qquad (h \in \mathcal{H}), \tag{17}$$

where $p_h^C C_h$ is nominal consumer spending in country h.

2. Input cost share matrix $\tilde{\Omega}$. Using the partition above, we construct a cost-based input-output matrix $\tilde{\Omega}$, where $\tilde{\Omega}_{ij}$ is the share of costs of i that comes from j, ordered with consumption goods first, followed by intermediate goods, investment goods, capital goods, and labor.[12] The structure of $\tilde{\Omega}$ is depicted below:

$$\tilde{\Omega} = \begin{pmatrix} \mathbf{0}_{H,H} & \tilde{\Omega}_{H,HN} & \mathbf{0}_{H,HX} & \mathbf{0}_{H,HK} & \mathbf{0}_{H,HF} \\ \mathbf{0}_{HN,H} & \tilde{\Omega}_{HN,HN} & \mathbf{0}_{HN,HX} & \tilde{\Omega}_{HN,HK} & \tilde{\Omega}_{HN,HF} \\ \mathbf{0}_{HX,H} & \tilde{\Omega}_{HX,HN} & \mathbf{0}_{HX,HX} & \mathbf{0}_{HX,HK} & \mathbf{0}_{HX,HF} \\ \mathbf{0}_{HK,H} & \mathbf{0}_{HK,HN} & \mathbf{I}_{HK,HX} & \mathbf{0}_{HK,HK} & \mathbf{0}_{HK,HF} \\ \mathbf{0}_{HF,H} & \mathbf{0}_{HF,HN} & \mathbf{0}_{HF,HX} & \mathbf{0}_{HF,HK} & \mathbf{0}_{HF,HF} \end{pmatrix}. \tag{18}$$

Equation (18) defines a square matrix and displays the submatrices that are non-zero. In the first line, the only non-zero block is $[\tilde{\Omega}_{H,HN}]_{h,h'i'}$, which is the share of household consumption expenditure in h that is spent on industry i' in country h'. The second line contains cost shares of intermediate goods $\mathcal{N}$. The first non-zero matrix $[\tilde{\Omega}_{HN,HN}]_{hi,h'i'}$ records the expenditures on intermediate inputs i' from h' relative to the total costs of the intermediate good producer i in country h. The subsequent matrices $[\tilde{\Omega}_{HN,HK}]_{hi,hk}$ and $[\tilde{\Omega}_{HN,HF}]_{hi,hf}$ do the same for rental payments to capital goods k and payments to labor

[12]Formally, $\tilde{\Omega}$ is a square matrix with dimension $|\mathcal{H}|(1+|\mathcal{N}|+2|\mathcal{K}|+|\mathcal{F}|)$, made up of $|\mathcal{H}|$ consumption goods, $|\mathcal{H}||\mathcal{N}|$ intermediate goods, $|\mathcal{H}||\mathcal{K}|$ investment goods, $|\mathcal{H}||\mathcal{K}|$ capital goods, and $|\mathcal{H}||\mathcal{F}|$ labor types; within each category, elements are ordered by type and then by country.

f for industry i in country h. On the third line, the only non-zero matrix $[\tilde{\Omega}_{HX,HN}]_{hk,h'i'}$ is the investment flow matrix measuring spending on inputs from industry i' in country h' as a share of total investment expenditures on capital good k in country h. On the fourth line, the only non-zero block is the identity matrix $\mathbf{I}_{HK,HX}$ which captures that all inputs to capital good k in country h come from the associated investment good in that country. The fifth line is all zeros reflecting that labor is a primary factor endowment that does not have any inputs. Every row apart from the rows with labor adds up to one.

3. Initial wedges. We define a vector of initial wedges with the same length as $\tilde{\Omega}$. The structure is given by

$$\boldsymbol{\mu} = [\underbrace{\mathbf{1}_H}_{\text{Consumption goods}}, \underbrace{\boldsymbol{\mu}^Y}_{\text{Intermediates}}, \underbrace{\mathbf{1}_{HK}}_{\text{Investment}}, \underbrace{\boldsymbol{\mu}^K}_{\text{Capital}}, \underbrace{\mathbf{1}_{HF}}_{\text{Labor}}],$$

with every element equal to one except those corresponding to intermediate goods, $\boldsymbol{\mu}^Y$, and capital goods, $\boldsymbol{\mu}^K$. Following (7), the intermediate good wedge, μ^Y_{hi}, is the ratio between the industry's total revenue and the total cost (inclusive of the user cost of capital). The capital wedge, $\mu^K_{hk} = R_{hk}K_{hk}/p^X_{hk}X_{hk}$, is the ratio between rental payments and investment expenditure on capital good k in country h.

Together, the expenditure shares Φ_h, the input cost share matrix $\tilde{\Omega}$ and the wedges $\boldsymbol{\mu}$ pin down industry sizes. Let λ be a vector of the same length as $\tilde{\Omega}$, where the a-th element λ_a is the sales of industry a relative to total nominal world consumption. When a is an investment good, sales are investment expenditures; when a is a capital good, sales are rental income $R_{hk}K_{hk}$; when a is a labor input, sales are labor compensation $w_{hf}L_{hf}$. The resource constraints (10)-(14) expressed in nominal units imply that the vector of sales shares $\boldsymbol{\lambda}$ solves the following equation:

$$\lambda_a = \Phi_a + \sum_b \frac{\lambda_b}{\mu_b}\tilde{\Omega}_{b,a}, \tag{19}$$

where we define $\Phi_a = 0$ when a is not a consumption good.

4. Elasticities of substitution. Let σ be an array that, for each industry, collects the elasticities of substitution between its inputs. For each producer a, we define $\sigma^a_{b,c}$ as the Allen-Uzawa elasticity of substitution between its inputs b and c. Denote the cost-weighted average elasticity of substitution between b and every other input by $\bar{\sigma}^a_b =$

$\sum_{c \neq b} \tilde{\Omega}_{ac} \sigma^a_{bc} / (1 - \tilde{\Omega}_{ab})$ for $\tilde{\Omega}_{ab} \in (0, 1)$.[13]

4.3 Characterization

Using the objects defined above, it is possible to characterize long-run comparative statics to first order. To state the result compactly, we collect non-labor prices into a single vector p, stacking consumption-good prices p^C_h, intermediate goods prices p^Y_{hi}, investment good prices p^X_{hk}, and capital rental rates R_{hk}. Wages w are kept separate.

Proposition 3 (Balanced Growth Path Comparative Statics). *Given an initial balanced growth path and perturbations $(d \log \mathbf{A}, d \log \boldsymbol{\mu})$, the first-order changes in balanced growth quantities and prices solve the linear system*

$$d \log p_a = -d \log A_a + d \log \mu_a + \sum_b \tilde{\Omega}_{a,b} \, d \log p_b + \sum_c \tilde{\Omega}_{a,c} d \log w_c, \tag{20}$$

$$\begin{aligned} d \log \tilde{\Omega}_{ab} &= -(1 - \tilde{\Omega}_{ab}) [\bar{\sigma}^a_b - 1] d \log p_b \\ &\quad + \sum_{d \neq b} \tilde{\Omega}_{ad} \, [\sigma^a_{bd} - 1] \, d \log p_d + \sum_{c \neq b} \tilde{\Omega}_{ac} \, [\sigma^a_{bc} - 1] \, d \log w_c, \end{aligned} \tag{21}$$

$$d\lambda_a = d\Phi_a + \sum_b \frac{\lambda_b}{\mu_b} \tilde{\Omega}_{ba} \left[d \log \lambda_b + d \log \tilde{\Omega}_{ba} - d \log \mu_b \right], \tag{22}$$

$$d\Phi_h = \sum_{f \in \mathcal{F}} \lambda_{L_{hf}} \, d \log w_{hf} + \sum_{k \in \mathcal{K}} d \left[\lambda_{K_{hk}} \left(1 - \frac{1}{\mu_{K_{hk}}} \right) \right] + \sum_{i \in \mathcal{N}} d \left[\lambda_{Y_{hi}} \left(1 - \frac{1}{\mu_{Y_{hi}}} \right) \right], \tag{23}$$

$$0 = d \log \lambda_{L_{hf}} - d \log w_{hf}, \tag{24}$$

where the price normalization is $d \log (\sum_h p^C_h C_h) = 0$. All coefficients in the system are pinned down by the expenditure shares Φ_h and $\tilde{\Omega}$, the elasticities of substitution $\boldsymbol{\sigma}$, and the initial wedges $\boldsymbol{\mu}$. Changes in capital wedges satisfy $d \log \mu_{K_{hk}} = \frac{d\tau^K_{hk}}{\delta_k + r_{hk}}$. The change in the quantity produced by a is $d \log \lambda_a - d \log p_a$.

The equation system in the proposition determines all endogenous variables in terms of shocks and the sufficient statistics defined above.[14] Equation (20) follows from Shephard's lemma. Equation (21) is a log-linearized input demand curve. Equation (22) is a

[13] In terms of model objects, $\sigma^a_{b,c} = \frac{\mathcal{C}_a \left[\partial^2 \mathcal{C}_a / \partial p_b \partial p_c \right]}{[\partial \mathcal{C}_a / \partial p_b][\partial \mathcal{C}_a / \partial p_c]}$, where $\mathcal{C}_a$ is the cost function of producer a. Note that we need not specify the Allen-Uzawa elasticity for pairs where one of the inputs is never used in that industry. In models with finite choke prices, comparative statics need to be defined in terms of derivatives rather than elasticities.

[14] In particular, the first four equations can be used to solve for everything in terms of shocks and changes in wages: (20)-(21) pin down changes in prices and factor shares from changes in wages; given these, (22)-(23) pin down changes in industry sizes $d \log \lambda_a$. Given this, the last equation (24) pins down wages.

log-linearized version of the market clearing conditions (19). Equation (23) is a linearization of the household budget constraint (2), normalized by aggregate world consumption. The last equation (24) follows from labor market clearing $d\log L_{hf} = 0$, since our choice of world consumption as numeraire, $\sum_h p_h^C C_h = 1$, implies $d\log L_{hf} = d\log(\lambda_{L_{hf}}/w_{hf})$. These equations together determine changes in prices $d\log p_a$ and sales shares $d\log\lambda_a$. Since changes in quantities are given by $d\log(\lambda_a/p_a)$, these follow directly. Proposition 3 pins down changes in all equilibrium variables to first order. However, as we discuss in Appendix F.1, it can also be used to calculate nonlinear comparative statics by chaining together first-order comparative statics, updating variables along the way, and cumulating, provided that we know how to update the elasticities of substitution.

5 Global Consumption Responses

We now use Propositions 2 and 3 to characterize long-run changes in global consumption. Changes in global consumption are easier to think through than local consumption responses because terms-of-trade effects cancel out for the world as a whole. Section 6 characterizes country-level consumption changes, which include terms-of-trade effects.

The change in global consumption is defined as the expenditure-weighted change in country-level consumption, analogous to how national accounts define real consumption. When there is only one country, global and national consumption coincide, which means that all our results for global consumption remain true for country-level consumption in a closed economy.

Definition 2 (Change in Global Consumption)**.** Given a change in parameters, the resulting change in global consumption is defined to be

$$d\log C = \sum_{h\in\mathcal{H}} \Phi_h \, d\log C_h,$$

where $\Phi_h = p_h^C C_h / \sum_{h'} p_{h'}^C C_{h'}$ is country h's share of global consumption, and $d\log C_h$ is the change in real consumption in country h.

We characterize the responses of global consumption to changes in productivities A_{hi}^Y, changes in wedges μ_{hi}^Y, and changes in wealth taxes τ_{hk}^K. An important special case is when the economy is at the Golden Rule, in which case a version of Hulten's theorem applies.

5.1 The Effect of Productivity Changes

To characterize the global consumption effect of changes in industry productivities A^Y_{hi}, we define the *cost weight vector* $\tilde{\lambda}$ as the solution to the following equation

$$\tilde{\lambda}_a = \Phi_a + \sum_b \tilde{\lambda}_b \tilde{\Omega}_{b,a}$$

where $\tilde{\Omega}$ is the input cost matrix and Φ is a vector of consumption shares.[15] Intuitively, the vector $\tilde{\lambda}$ captures the importance of industries and factors to the cost of producing the global consumption bundle on the balanced growth path, directly and indirectly through the production and investment network.[16] Note that if there are no wedges, $\boldsymbol{\mu} = \mathbf{1}$, then the cost weights $\tilde{\lambda}_a$ coincide with industry sales λ_a as in (19).

Proposition 4 (Productivity Shocks)**.** *The long-run elasticity of global consumption with respect to a productivity change in sector hi is*

$$\frac{\partial \log C}{\partial \log A^Y_{hi}} = \tilde{\lambda}_{Y_{hi}} - \sum_{h \in \mathcal{H}, f \in \mathcal{F}} \tilde{\lambda}_{L_{hf}} \frac{\partial \log \lambda_{L_{hf}}}{\partial \log A^Y_{hi}},$$

where $\lambda_{L_{hf}} = w_{hf} L_{hf} / \sum_{h'} p^C_{h'} C_{h'}$ is the compensation to labor of type f in country h as a share of world consumption, with $d \log \lambda_{L_{hf}}$ characterized in Proposition 3.

The two summands in Proposition 4 are interpretable. The first term $\tilde{\lambda}_{Y_{hi}}$ captures the mechanical *technological* effect of changing A^Y_{hi} and the second term captures a general-equilibrium *reallocation* effect. We discuss these two mechanisms in turn.

Technological effects. The first term $\tilde{\lambda}_{Y_{hi}}$ is the technological effect of A^Y_{hi}, defined as the effect of changing A^Y_{hi} keeping the allocation of resources constant. More precisely, suppose A^Y_{hi} rises and that every industry continues to allocate its output across downstream users in the same proportions as before. In that case, the increase in consumption comes from the output in *hi* flowing through the existing input-output and investment networks, feeding the direct users of *hi*, the second-degree users, and so on. The resulting increase in global consumption is given by $\tilde{\lambda}_{Y_{hi}}$. We call this the technological effect of A^Y_{hi} since it obtains when reallocation is absent (see Appendix C.1 for formal definitions and proofs, and Baqaee and Farhi (2020) for further discussion).

[15] In vector form, $\tilde{\lambda}' = \Phi'(I - \tilde{\Omega})^{-1}$; recall that Φ is padded with zeros to conform with $\tilde{\Omega}$, with only consumption goods having positive weight.

[16] Formally, equation (20) in Proposition 3 can be expressed as $\sum_h \Phi_h d \log p^C_h = -\sum_{h \in \mathcal{H}, i \in \mathcal{N}} \tilde{\lambda}_{Y_{hi}} d \log A^Y_{hi} + \sum_{h \in \mathcal{H}, f \in \mathcal{F}} \tilde{\lambda}_{L_{hf}} d \log w_{hf}$, which means that $\tilde{\lambda}$ measures the elasticity of the cost of the global consumption bundle to productivities and factor prices.

When all production technologies and consumption aggregators are Cobb-Douglas, the equilibrium endogenously features no reallocation, and the following corollary obtains.

Corollary 1 (Cost weights as sufficient statistics)**.** *If every production function and consumption aggregator is Cobb-Douglas, then the long-run elasticity of global consumption with respect to a change in the productivity A^Y_{hi} is given by:*

$$\frac{\partial \log C}{\partial \log A^Y_{hi}} = \tilde{\lambda}_{Y_{hi}}.$$

That is, under Cobb-Douglas, the cost weight $\tilde{\lambda}_{Y_{hi}}$ gives the full consumption effect of productivity.[17] This corollary follows immediately from Propositions 3 and 4 using the fact that, when all aggregators are Cobb-Douglas, every elasticity of substitution $\sigma^a_{bc} = 1$.

Note that $\tilde{\lambda}_{Y_{hi}}$ is constructed from input cost shares and consumption shares alone, without the array of elasticities of substitution. In Section 8, we show that the cost weights $\tilde{\lambda}$ provide a very good approximation of the full global consumption effect in a calibrated world economy, even when production functions are not exactly Cobb-Douglas and Corollary 1 does not apply.

Importantly, the cost weight $\tilde{\lambda}_{Y_{hi}}$ of an industry is *not* simply proportional to the industry's sales, unless the economy is at the Golden Rule. In general, the cost weight places greater weight on investment good producers and on the industries that supply them, directly or indirectly. The reason is that whenever capital is below its Golden Rule level, we have $\boldsymbol{\mu}^K > 1$, and the sales of capital good sectors $R_{hk}K_{hk}$ exceed their investment spending $p^X_{hk}X_{hk}$. This depresses sales measures for investment good sectors and their upstream suppliers. The following example illustrates.

Example 1 (Cost weights and sales in a chain economy)**.** Consider an economy where final consumption is produced from labor via a chain of capital goods. Labor is used to produce an investment good X_1, which is used to accumulate a capital good K_1, which in turn is used to produce another investment good X_2, and so forth, with capital good K_n used to produce the final consumption good (see Figure 2). We assume a constant growth rate g and a common depreciation rate δ.

[17] The absence of reallocation under Cobb-Douglas can be understood in two ways. Through the lens of Proposition 4, Cobb-Douglas implies that labor income shares $\lambda_{L_{hf}}$ are constant, which means that the reallocation term $-\sum_{h\in\mathcal{H},f\in\mathcal{F}} \tilde{\lambda}_{L_{hf}} \frac{\partial \log \lambda_{L_{hf}}}{\partial \log A^Y_{hi}}$ vanishes. Alternatively, the lack of reallocation can be understood directly by noting that the allocation of an industry a's output across different users is determined by the relative sales of those users and input a's share of those sales. Since neither moves with A^Y_{hi} under Cobb-Douglas, the distribution across downstream users is fixed, so reallocation is absent.

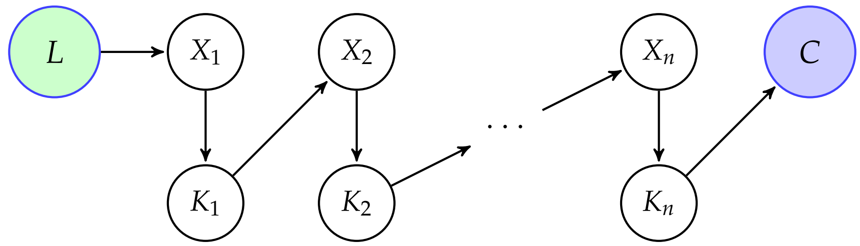

Figure 2: Labor produces consumption through a chain of capital goods

In the equivalent static economy, all input costs paid by an industry go to the previous link in the chain. The input cost matrix $\tilde{\Omega}$ satisfies

$$\tilde{\Omega}_{C,K_n} = \tilde{\Omega}_{K_i,X_i} = \tilde{\Omega}_{X_i,K_{i-1}} = \cdots = \tilde{\Omega}_{X_1,L} = 1,$$

with all other elements being zero.

The structure of $\tilde{\Omega}$ means that each step of the chain has the same cost weight: $\tilde{\lambda}_a = 1$. To see this, note that for K_n, the capital good closest to final consumption, this follows from $\tilde{\lambda}_{K_n} = \Phi_C = 1$. For X_n, the investment good closest to final consumption, we have $\tilde{\lambda}_{X_n} = \tilde{\lambda}_{K_n}\tilde{\Omega}_{K_n,X_n} = \tilde{\lambda}_{K_n} = 1$, and so on for each upstream industry, ending with $\tilde{\lambda}_L = 1$.

In contrast, the sales weights λ_a become progressively smaller as we move upstream in the chain. For the capital good K_n closest to final consumption, we still have $\lambda_{K_n} = \Phi_C = 1$. However, for X_n, the investment good closest to consumption, we have $\lambda_{X_n} = (\lambda_{K_n}/\mu_{K_n})\tilde{\Omega}_{K_n,X_n} = [(r+\delta)/(g+\delta)]^{-1}$, which is smaller than one whenever $r > g$ and the rental payments to K_n exceed the investment expenditure on X_n. The same logic means that each time we pass through a capital good, a fraction of sales becomes net capital income and does not reach the investment good supplier. We obtain the following sales shares:

$$\lambda_{K_i} = \left(\frac{r+\delta}{g+\delta}\right)^{-(n-i)} \qquad \lambda_L = \left(\frac{r+\delta}{g+\delta}\right)^{-n}.$$

That is, whenever $r > g$, the sales shares of upstream investment goods and labor are strictly smaller than their cost shares.

The pattern in Example 1, with investment goods and their suppliers having low sales weights, remains true in realistic production and investment networks. Table 1 displays sales and cost-based weights for selected industries, aggregated across countries (Section 8 describes how we measure these objects from world input-output data and investment tables). As in the example, the gaps are largest for industries that produce investment goods, such as construction, and for industries that lie upstream from investment

Table 1: Global sales and cost weights for selected industries

Industry	**World sales weight** $\sum_h \lambda_{Y_{hi}}$	**World cost-based weight** $\sum_h \tilde{\lambda}_{Y_{hi}}$
Construction	0.135	0.379
Agriculture	0.083	0.095
Basic and fabricated metals	0.082	0.184
Machinery	0.047	0.125
Textiles	0.034	0.039
Food Products	0.109	0.114

Notes: The first column gives the sum of sales of industries of type i across all countries, expressed relative to world consumption. The second column reports the sum of cost weights. The weights are averaged over the period 1995–2009 and are constructed from the World Input–Output Database (Timmer et al., 2015) and the global investment-flow table of Ding (2022). Details on the construction are provided in Section 8.

goods, such as basic metals. For industries that are not upstream of investment goods, like agriculture and textiles, sales weights are similar to cost weights.

Reallocation effects. The second term in Proposition 4, $-\sum_{h\in\mathcal{H},f\in\mathcal{F}} \tilde{\lambda}_{L_{hf}} \frac{\partial \log \lambda_{L_{hf}}}{\partial \log A^{Y}_{hi}}$, is a *reallocation effect*. When production functions are not Cobb-Douglas, changes in productivity, A^{Y}_{hi}, trigger reallocations across downstream uses through changes in relative prices. These reallocations generally have a first-order effect on global consumption whenever the initial allocation does not maximize long-run consumption. The proposition shows that the reallocation effect is summarized by a weighted decrease in labor shares: intuitively, labor shares fall when a reallocation moves resources toward uses with higher wedges. If the only wedges are the as-if capital wedges μ^K, this is equivalent to the reallocation increasing the capital intensity of the economy. The following example illustrates.

Example 2 (Reallocation effects)**.** Consider a neoclassical growth model where output is produced using a CES production function:

$$Y = A_Y \left[(1-\alpha)^{\frac{1}{\sigma_{KL}}} L^{\frac{\sigma_{KL}-1}{\sigma_{KL}}} + \alpha^{\frac{1}{\sigma_{KL}}} K^{\frac{\sigma_{KL}-1}{\sigma_{KL}}} \right]^{\frac{\sigma_{KL}}{\sigma_{KL}-1}},$$

where A_Y is the TFP of final good production. Proposition 4 implies that the long-run effect on consumption from changing A_Y is given by

$$\frac{d\log C}{d\log A_Y} = \overbrace{\frac{1}{1-\alpha}}^{\tilde{\lambda}_Y} + \overbrace{(\sigma_{KL}-1)\frac{\lambda_X}{1-\alpha}\frac{r-g}{g+\delta}}^{-\tilde{\lambda}_L \frac{d\log\lambda_L}{d\log A^Y}},$$

where $\lambda_X = \frac{p^X X}{p^C C}$ is the sales of the investment good sector relative to consumption, and where initial prices are normalized to one so that α equals the capital share $\tilde{\Omega}_{Y,K}$ (see Appendix C.2 for the derivation).

The first term in the expression is the standard neoclassical amplification term $1/(1-\alpha)$, which is the cost weight $\tilde{\lambda}_Y$ of the final good sector. Under Cobb-Douglas ($\sigma_{KL} = 1$), this is the full effect, in line with Corollary 1. The second term is the reallocation channel, which is only active when there is a joint deviation from Cobb-Douglas ($\sigma_{KL} \neq 1$) and the Golden Rule ($r \neq g$). For instance, if capital and labor are gross complements ($\sigma_{KL} < 1$) and $r > g$, then an increase in productivity lowers capital intensity and increases the labor share ($d\log\lambda_L > 0$). This reallocation dampens the increase in long-run global consumption since the initial capital stock was below its Golden Rule value.

5.2 The Effect of Changes in Wedges and Returns

Next, we consider the effect of changes in distortions $\mu_{Y_{hi}}$ and wealth taxes τ^K_{hk}. Both of these effects operate through changes in the vector of wedges $\boldsymbol{\mu}$ in the equivalent static economy. Since such changes have no effect on technology, they only affect long-run consumption by changing the allocation of resources. This effect depends on the interaction of initial wedges with changes in quantities. The following proposition writes the consumption response in terms of a Harberger (1964)-style formula.

Proposition 5 (Changes in Distortions)**.** *The first-order effect on global real consumption from a change in a wedge $d\log\mu_a$ satisfies:*

$$\frac{d\log C}{d\log\mu_a} = \sum_{h\in\mathcal{H},i\in\mathcal{N}} \lambda_{Y_{hi}}\left(1-\frac{1}{\mu_{Y_{hi}}}\right)\frac{d\log Y_{hi}}{d\log\mu_a} + \sum_{h\in\mathcal{H},k\in\mathcal{K}} \lambda_{K_{hk}}\left(\frac{r_{hk}-g}{r_{hk}+\delta_k}\right)\frac{d\log K_{hk}}{d\log\mu_a}. \quad (25)$$

Shocks to intermediate input wedges $d\log\mu_{Y_{hi}}$ are primitive shocks, and shocks to capital wedges are given by $d\log\mu_{K_{hk}} = \frac{d\tau^K_{hk}}{\delta_k + r_{hk}}$.

The first term in the proposition is the classic formula from Harberger (1964): consumption rises if quantities rise in markets with initially high wedges. The twist in dynamic economies is that as-if capital wedges $1 - 1/\mu^K_{hk} = \frac{r_{hk}-g}{r_{hk}+\delta_k}$ play a similar role to genuine distortions for long-run consumption. This means that changes in wedges also raise aggregate consumption to the extent that they raise capital stocks, with the effect scaling with the distance to the Golden Rule, $r_{hk} - g$.

For changes in capital taxes τ^K_{hk}, note that these only affect long-run outcomes through rates of return r_{hk}, which by Proposition 2 only matter through changes in the capital wedges $\mu_{K_{hk}}$. Therefore, the effect of changes in capital taxes follows the same logic as changes in production wedges $\mu_{Y_{hi}}$.

One implication of Proposition 5 is that changes in taxes typically have a first-order effect on long-run consumption even if the initial dynamic economy is undistorted. The reason is that away from the Golden Rule, anything that changes capital stocks changes long-run consumption to first order. The standard envelope theorem argument for zero first-order effects does not apply to long-run consumption, since it is typically not maximized in an undistorted dynamic economy. The following example illustrates this point for the case of tariffs.

Example 3 (Effects of a Tariff)**.** Consider a model with two symmetric countries, home h and foreign f. Economy h has the following balanced growth equations (economy f is identical with switched indices):

$$\begin{aligned}
C_h + X_h &= \left((1-\beta)^{\frac{1}{\theta}} Y_{h,h}^{\frac{\theta-1}{\theta}} + \beta^{\frac{1}{\theta}} Y_{h,f}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}, \\
Y_h &= \left(\alpha^{\frac{1}{\sigma_{KL}}} K_h^{\frac{\sigma_{KL}-1}{\sigma_{KL}}} + (1-\alpha)^{\frac{1}{\sigma_{KL}}} L_h^{\frac{\sigma_{KL}-1}{\sigma_{KL}}} \right)^{\frac{\sigma_{KL}}{\sigma_{KL}-1}} \\
Y_h &= Y_{h,h} + Y_{f,h}. \\
X_h &= (g+\delta) K_h
\end{aligned}$$

The first equation states that consumption and investment in h use the same Armington aggregate of domestic $Y_{h,h}$ and foreign goods $Y_{h,f}$ with elasticity of substitution θ. The second equation states that h produces its variety using capital and labor; the next equation shows that the variety is used either domestically or exported. The final equation is the capital accumulation equation. We normalize initial prices to one, so that on the initial balanced growth path, β corresponds to the foreign share of final output and α to the capital share of GDP.

Starting from no tariffs, we introduce a two-way symmetric tariff, t, that is rebated to domestic households. Propositions 3 and 5 imply that the long-run change in global consumption is given by

$$d\log C = \lambda_K \frac{r-g}{r+\delta} d\log K = -\lambda_K \frac{r-g}{r+\delta}\sigma_{KL}\frac{\beta}{1-\alpha} d\log(1+t),$$

where we suppress the country indices because they are symmetric. When the economy is not at the Golden Rule, $r \neq g$, a tariff has a first-order effect on long-run consumption. The mechanism is that tariffs depress the capital stock by making investment goods more expensive relative to labor. The effect's magnitude depends positively on the import share β, the capital share α, and the elasticity between capital and labor σ_{KL}. However, the effect does not depend on the Armington trade elasticity θ, which is the key determinant of the cost of tariffs in static trade models. In a static trade model $d\log K = 0$, and there are no first-order losses from a tariff.

5.3 The Golden Rule Benchmark

For all comparative statics, the Golden Rule plays the role of a neutral benchmark. When $r_{hk} = g$ and there are no distortions, $\mu_{Y_{hi}} = 1$, the equivalent static economy is efficient, and the long-run comparative statics simplify considerably.

Proposition 6 (Golden Rule Benchmark)**.** *Suppose that the economy is at the Golden Rule ($r_{hk} = g$ for all capital goods) and there are no distortions, $\mu_{Y_{hi}} = 1$. Then, the equivalent static economy is undistorted, and long-run comparative statics are given by:*

$$\frac{d\log C}{d\log A^Y_{hi}} = \lambda_{Y_{hi}}, \qquad \frac{d\log C}{d\log \mu_{Y_{hi}}} = 0, \qquad \frac{d\log C}{d\tau^K_{hk}} = 0. \tag{26}$$

The result shows the central role of deviations from the Golden Rule in making long-run comparative statics differ from their static counterparts. At the Golden Rule, results from undistorted static economies carry over almost directly. For productivity, effects are equal to sales shares $\lambda_{Y_{hi}}$, with no role for elasticities or for the network structure beyond what is encoded in sales shares, just as in Hulten (1978). The one difference from Hulten is that sales are measured relative to consumption rather than to GDP, consistent with capital being an intermediate input from a balanced growth perspective. Furthermore, since the Golden Rule economy maximizes long-run consumption, the envelope theorem applies and there are no first-order effects from changes in wedges $\mu_{Y_{hi}}$. The same applies to changes in rates of return coming from changes in capital taxes τ^K_{hk}, since these operate exclusively by changing wedges.

5.4 Gross Domestic Product and Development Accounting

Until now, the outcome of interest has been consumption. In applied work, we are also often interested in GDP, which includes investment in addition to consumption. The change in real GDP is defined to be the expenditure-weighted change in consumption and investment:

$$d \log Y = \sum_{h \in \mathcal{H}} \frac{p_h^C C_h}{E} d \log C_h + \sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}} \frac{p_{hk}^X X_{hk}}{E} d \log X_{hk},$$

where $E \equiv \sum_{h \in \mathcal{H}} p_h^C C_h \ + \ \sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}} p_{hk}^X X_{hk}$ is nominal GDP. This can also be computed from Proposition 3. To get more intuition, we rewrite changes in aggregate GDP as

$$d \log Y = d \log C + \frac{s_X}{1 - s_X} d \log \frac{X}{Y}, \tag{27}$$

where $s_X = \sum_{h,k} p_{hk}^X X_{hk} / E$ is the investment share of GDP and $d \log X = \sum_{h,k} \Phi_{hk}^X d \log X_{hk}$ is the change in aggregate real investment, with $\Phi_{hk}^X = \frac{p_{hk}^X X_{hk}}{\sum_{h' \in \mathcal{H}, k' \in \mathcal{K}} p_{h'k'}^X X_{h'k'}}$ being the expenditure on investment good k in country h as a share of aggregate nominal investment.[18]

Given a shock to productivity A_{hi}^Y, the long-run GDP effect is given by the change in aggregate consumption plus the change in the investment-to-GDP ratio, scaled by the ratio of investment to consumption expenditure. For consumption, we know from Proposition 4 and Corollary 1 that the effect of A_{hi}^Y depends on the exposure of consumption costs to industry hi, summarized by the cost weight $\tilde{\lambda}_{Y_{hi}}$. For investment-to-output, the effect of A_{hi}^Y depends on the *relative* exposure of investment and consumption to changes in A_{hi}^Y. In the case with Cobb-Douglas production functions and preferences, the result is particularly sharp, as shown in the following proposition (see Appendix C.4 for the general expression).

Proposition 7 (Productivity Shocks and GDP)**.** *If every production function and consumption aggregator is Cobb-Douglas, the change in investment relative to output given a change in productivity A_{hi}^Y is $\frac{d \log X/Y}{d \log A_{hi}^Y} = (1 - s_X) \left(\tilde{\lambda}_{Y_{hi}}^X - \tilde{\lambda}_{Y_{hi}} \right)$. The long-run effect on real GDP is*

$$\frac{d \log Y}{d \log A_{hi}^Y} = \tilde{\lambda}_{Y_{hi}} + s_X \left(\tilde{\lambda}_{Y_{hi}}^X - \tilde{\lambda}_{Y_{hi}} \right),$$

where $\tilde{\lambda}_{Y_{hi}}^X = \sum_{h \in \mathcal{H}, k \in \mathcal{K}} \Phi_{hk}^X (I - \tilde{\Omega})_{hk,hi}^{-1}$ is the investment analogue of the cost weight $\tilde{\lambda}$.

[18] To derive the expression, note that $d \log Y = (1 - s_X) d \log C + s_X d \log X$, subtract $s_X d \log Y$ and divide by $1 - s_X$.

The proposition provides a recipe for predicting the GDP effect of productivity changes, including the indirect effect on capital accumulation. Use the input-output and investment network to calculate the investment share of GDP s_X as well as the consumption and investment cost weights $\tilde{\lambda}_{Y_{hi}}$ and $\tilde{\lambda}^X_{Yhi}$ of hi. The cost weight $\tilde{\lambda}_{Yhi}$ gives the consumption effect by Corollary 1, while the change in the investment-to-output ratio is given by $(1 - s_X)(\tilde{\lambda}^X_{Yhi} - \tilde{\lambda}_{Yhi})$, with the effect weighted by s_X.

One implication of the proposition is that since changes in productivity can move the ratio of investment to output, X/Y, they can also move the capital-to-output ratio, K/Y. This is important for the development accounting literature, where it is common to interpret differences in K/Y as reflecting the propensity to accumulate, and to interpret differences in output per worker conditional on K/Y as TFP (Klenow and Rodriguez-Clare, 1997; Hall and Jones, 1999). The decomposition is natural when productivity does not affect K/Y, as with TFP shocks in a standard neoclassical growth model. It is less natural when K/Y moves with productivity. The following example illustrates this point by considering consumption-specific and investment-specific productivity changes in the neoclassical growth model.

Example 4 (GDP in the neoclassical growth model)**.** Consider a neoclassical growth model with a Cobb-Douglas aggregate production function

$$Y = A_Y K^{\alpha} (A_L L)^{1-\alpha}.$$

Furthermore, let A_C be a productivity shifter for converting final goods into consumption and A_X a shifter for converting final goods into investment. The table reports the cost weights of consumption and investment, $\tilde{\lambda}$ and $\tilde{\lambda}^X$, for each productivity shifter, together with the comparative statics of consumption $d \log C$, the change in the capital-to-output ratio $d \log K/Y$ (which is the same as $d \log X/Y$), and the change in real GDP $d \log Y$.

Labor-augmenting and Hicks-neutral productivities A_L and A_Y affect consumption and investment equally: K/Y does not move, and GDP moves one-for-one with consumption. Investment-specific productivity A_X affects investment more: K/Y rises, and GDP rises by more than consumption. Consumption-specific A_C productivity does not affect investment: K/Y falls, and GDP rises by less than consumption.

The example shows that the neutrality of K/Y with respect to productivity in the standard neoclassical growth model reflects a particular model feature: consumption and investment are equally exposed to productivity changes. When this coincidence breaks, changes in productivity move K/Y, and output given K/Y no longer isolates the effect of TFP. For example, if the technological advantage of rich countries is investment-biased —

Table 2: Cost weights and comparative statics in the neoclassical growth model

	A_L	A_Y	A_X	A_C
$\tilde{\lambda}$	1	$\frac{1}{1-\alpha}$	$\frac{\alpha}{1-\alpha}$	1
$\tilde{\lambda}^X$	1	$\frac{1}{1-\alpha}$	$\frac{1}{1-\alpha}$	0
$d\log(K/Y)$	0	0	$1-s_X$	$-(1-s_X)$
$d\log C$	1	$\frac{1}{1-\alpha}$	$\frac{\alpha}{1-\alpha}$	1
$d\log Y$	1	$\frac{1}{1-\alpha}$	$\frac{\alpha}{1-\alpha}+s_X$	$1-s_X$

Notes: The $\tilde{\lambda}$ and $\tilde{\lambda}^X$ are the cost weights of consumption and investment associated with labor, final output, investment, and consumption.

as suggested by relative prices (Hsieh and Klenow, 2007) — some of the higher K/Y in rich countries should be attributed to technology.

6 Small Open Economy Approximation

Next, we study the response of long-run country-level consumption $d\log C_h$ to shocks. By definition, the change in consumption in country h is $d\log C_h = d\log\lambda_{C_h} - d\log p_{C_h}$, where $d\log\lambda_{C_h}$ and $d\log p_{C_h}$ are determined by Proposition 3. In general, these terms depend on the entire structure of the world economy even for a purely domestic shock (see Appendix D.1 for details). Intuitively, domestic changes affect prices and quantities in the rest of the world, which feed back into the domestic economy via changes to import prices and export demand schedules.

To cut through this complexity, we use a small open economy approximation to greatly simplify the analysis and reduce the information requirements. We first extend the equivalence result in Proposition 2 to small open economies. We then use the equivalence to characterize long-run comparative statics, focusing on an especially tractable special case where the response of domestic consumption to domestic productivity shocks, taking into account both long-run capital accumulation and changes in the terms of trade, boils down to just the small open economy analogue of the cost weights $\tilde{\lambda}^{soe}$.

6.1 Equivalence and Comparative Statics

Consider a country h in the model in Section 3 and impose balanced trade. Total exports of good i by country h are given by:

$$Y^{exp}_{hi} = \sum_{h'\neq h}\left[Y^C_{h',hi} + \sum_{i'\in\mathcal{N}} Y^Y_{h'i',hi} + \sum_{k\in\mathcal{K}} Y^X_{h'k,hi}\right].$$

This quantity equals the use of hi in foreign countries as consumption, intermediate inputs, and investment. A small open economy approximation for country h is defined by taking as given import prices and isoelastic export demand schedules:

$$p_{hi}^{Y} = D_{hi}^{\frac{1}{\theta}} \times (Y_{hi}^{exp})^{-\frac{1}{\theta}}, \tag{28}$$

where the D_{hi} are export-demand shifters and θ is the elasticity of export demand, with $\theta > 1$.[19]

The following result extends the equivalence between balanced growth paths and static economies from Proposition 2. We focus on the case with balanced trade; the proof in the appendix discusses how the results are modified by the presence of trade deficits.

Proposition 8 (Extending the Equivalence to Small Open Economies)**.** *Given country-level parameters $\boldsymbol{A}_h$, $\boldsymbol{\mu}_h^Y$, $\boldsymbol{\tau}_h^K$, import prices $\boldsymbol{p}_{-h}$, and export demand shifters $\boldsymbol{D}_h$, the balanced growth path prices and quantities of the dynamic small open economy without trade deficits also form an equilibrium of an equivalent economy that is static and closed. In this equivalent economy, (i) the wedges μ^Y and the production functions for consumption, intermediate, and investment goods are the same as in the dynamic economy; (ii) capital stocks K_{hk} are replaced by capital services that are produced instantaneously from investment goods according to the production function $K_{hk} = A_{hk}^K X_{hk}$, with productivity shifter $A_{hk}^K = 1/(g+\delta_k)$; (iii) capital services are sold at a markup $\mu_{hk}^K = (r_{hk}+\delta_k)/(g+\delta_k)$, with markup revenue from capital good k in country h distributed to households in that country and $r_{hk} = \rho + g/\gamma + \tau_{hk}^K$; (iv) there is an additional intermediate good – "dollar" – that is produced from exports using a production function with decreasing returns to scale $Y_{h\$} = \sum_{i\in\mathcal{N}} D_{hi}^{\frac{1}{\theta}} \times (Y_{hi}^{exp})^{1-\frac{1}{\theta}}$, and that is used to produce import goods using a linear technology with productivity $1/p_{h'j}^Y$ for imports of j from $h' \neq h$; (v) dollar production is subject to a subsidy wedge $\mu_{h\$}^Y = (1-1/\theta) \leq 1$.*

Conditions (i)-(iii) are identical to those in Proposition 2. Condition (iv) recasts trading opportunities as technologies: imports are "produced" using a dollar good, and the dollar good is produced using export goods. Crucially, the technology for producing dollars from exports has decreasing returns to scale — this captures the fact that export demand is not perfectly elastic. In the open economy, as more export goods are sold, the price of exports falls, and this lowers the marginal revenue of additional exports. In the equivalent closed economy, this same force is captured via decreasing returns to scale. Condition (v)

[19] Formally, we define a constant elasticity small open economy equilibrium by a vector of import prices $\{p_{h'i}^Y\}_{h'\neq h, i\in\mathcal{N}}$, the export demand schedules (28), and the BGP equations (2)-(16) associated with country h. See Appendix D.2 for a full definition, and Appendix D.4 for how to generalize the notion of a small open economy equilibrium to more flexible export demand systems.

states that a decentralized representation requires a wedge $\mu^{Y}_{h\$} = 1 - 1/\theta \leq 1$ that acts as an effective subsidy to export producers. This wedge captures that producers in the small open economy do not internalize terms-of-trade effects, which means that the small open economy is overexporting from the perspective of domestic consumption. The wedge on exports collapses to 1 when export demand is infinitely elastic, $\theta = \infty$, or when there is an optimal tariff in place.[20]

Since Proposition 8 converts a small open economy into an equivalent closed static economy, we can apply Proposition 3 to characterize long-run comparative statics for small open economies. In particular, using Proposition 3, we can express the comparative statics of small open economies in terms of the domestic input-output and investment flow tables, initial domestic wedges, domestic elasticities of substitution, and the export demand elasticity. The explicit construction of the equivalent static input-output matrix $\tilde{\Omega}^{soe}$, wedge vector $\boldsymbol{\mu}^{soe}$, and elasticity array σ^{soe} is given in Appendix D.5.

6.2 Useful Special Case

We now highlight a particularly tractable special case, where the elasticity of domestic consumption to a domestic productivity shock takes a very simple form.

Proposition 9 (Cobb-Douglas small open economy)**.** *Consider the small open economy of Proposition 8. Suppose that domestic consumption, production, and investment functions are Cobb-Douglas over domestic inputs and industry-specific foreign input aggregates. The composition of each foreign input aggregate may otherwise be unrestricted. Suppose, in addition, that sectoral exports are combined into a single export good with a Cobb-Douglas technology. Then, for every domestic industry i,*

$$\frac{\partial \log C_h}{\partial \log A^{Y}_{hi}} = \tilde{\lambda}^{soe}_{Y_{hi}},$$

where $\tilde{\lambda}^{soe}_h = \Phi^{soe}_h (I - \tilde{\Omega}^{soe}_h)^{-1}$.

Although the assumptions of Proposition 9 may not hold exactly in a disaggregated quantitative trade model (e.g., Costinot and Rodriguez-Clare, 2014), in Section 8, we show that $\tilde{\lambda}^{soe}$ nevertheless approximates the true response of long-run consumption very well in such models. This is useful, because $\tilde{\lambda}^{soe}$ can be obtained by a simple matrix inversion of the input cost matrix $\tilde{\Omega}^{\mathrm{soe}}$, without solving the full world equilibrium.[21]

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[20]The idea of treating small open economies as closed economies with a trading technology has a long history in the literature of international trade. For example, see the textbook treatment in Dixit and Norman (1980). Our proposition extends this equivalence to settings with downward-sloping export demand.

[21]To see why $\tilde{\lambda}^{\mathrm{soe}}_{Y_{hi}}$ is approximately equal to the long-run consumption response, note the following.

To see how the cost weight $\tilde{\lambda}^{\text{soe}}$ captures the effects of long-run capital accumulation and changes in the terms of trade, consider the following illustrative example.

Example 5 (Imports as Input to Investment)**.** Consider a small open economy where local consumption is produced using capital and labor. Capital is made from investment goods, which are made exclusively from imports. Exports are made directly from domestic labor. The equations for the production side of the economy are:

$$C_h = K_h^{\alpha} L_h^{1-\alpha}, \qquad \dot{K}_h = X_h - \delta K_h, \qquad X_h = F_h^X(\{Y_{h,h'}\}), \qquad Y^{\text{exp}} = A^{\text{exp}}(1 - L_h),$$

where the labor endowment is normalized to one. Hence, L_h is the share of labor employed by the consumption sector and $1 - L_h$ is the share employed by the export sector. Figure 3 illustrates the production network in the equivalent static and closed economy.

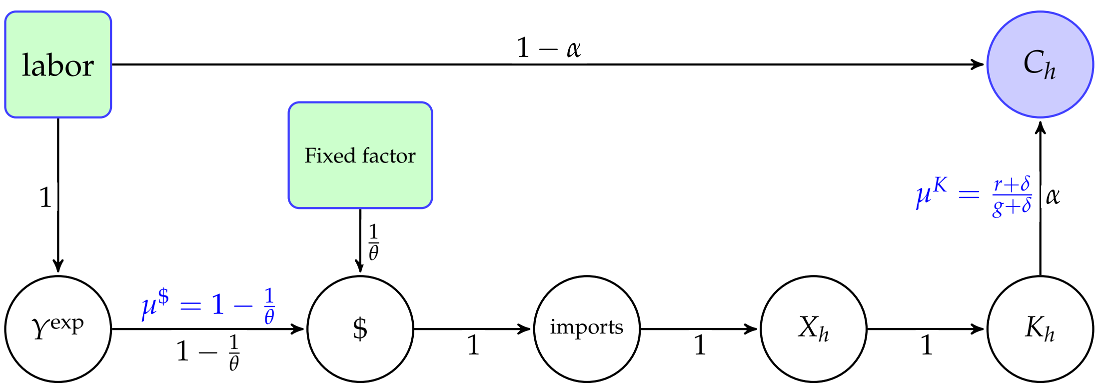

Figure 3: Arrows denote flows of goods; cost shares are in black and wedges in blue.

We are interested in the long-run consumption effects of an improvement in the productivity of the export sector, A^{exp}. Proposition 9 implies

$$\frac{\partial \log C_h}{\partial \log A^{\text{exp}}} = \tilde{\lambda}^{soe}_{Y^{\text{exp}}} = \mu^{\$}\mu^K \lambda_{Y^{\text{exp}}} = \frac{p^{\text{exp}} Y^{\text{exp}}}{p_h^C C_h}\left(1 - \frac{1}{\theta}\right)\left(\frac{r+\delta}{g+\delta}\right).$$

We see that the productivity effect only coincides with the sales weight of the export sector when export demand is infinitely elastic ($\theta = \infty$) and the economy is at the Golden Rule ($r = g$). In general, the cost and sales weights differ because of the wedges traversed on

By Proposition 4, the general response of long-run consumption to a domestic productivity shock under the open economy approximation is $\frac{\partial \log C_h}{\partial \log A^Y_{hi}} = \tilde{\lambda}^{soe}_{Y_{hi}} - \sum_{f \in \mathcal{F}} \tilde{\lambda}^{\text{soe}}_{L_{hf}} \frac{d \log \lambda^{soe}_{L_{hf}}}{d \log A^Y_{hi}} - \frac{\tilde{\lambda}^{\text{soe}}_{\$}}{\theta} \frac{d \log \lambda^{soe}_{\$}}{d \log A^Y_{hi}}$. In most standard quantitative trade models, aggregate labor income and export expenditures relative to consumption ($\lambda^{soe}_{L_{hf}}$ and $\lambda^{soe}_{\$}$) do not respond strongly to productivity shocks. Hence, the additional terms are quantitatively small. Proposition 9 provides conditions under which these terms are exactly zero.

the path from export production to final consumption. There are two such wedges. First, there is an as-if wedge $\mu^{\$} = 1 - 1/\theta$ between exports and the dollar good (to capture terms-of-trade effects), and second, there is an as-if wedge $\mu^K = (r+\delta)/(g+\delta)$ between the capital good and consumption (to capture deviations from the Golden Rule). The two effects work in opposite directions: inelastic export demand makes the export sector less important for long-run consumption relative to its sales; deviations from the Golden Rule make it more important.

7 Richer Models of Asset Accumulation

In this section, we relax the assumption that asset demand is infinitely elastic and that international capital markets are closed. We adjust our baseline model from Section 3 to have an alternative household savings block, allowing for three features: imperfectly elastic household savings, endogenously determined risk premia, and endogenous net foreign asset positions. The equivalence result from Proposition 2 carries over, together with auxiliary conditions that pin down changes in rates of return and net foreign asset positions from asset market clearing. The information required for comparative statics is the same as in Proposition 3, plus two new inputs: initial net foreign asset positions and asset demand elasticities with respect to returns.

Alternative model of household savings. We modify the model in Section 3 by changing the household savings block. Instead of countries having infinitely-lived representative agents, they are populated by overlapping generations of individuals, modeled using a perpetual youth setup in the style of Blanchard (1985) and Yaari (1965). In addition to investments in domestic capital goods B_{hk}, households also have access to an internationally traded bond B_{h0} with return r_0. Furthermore, investments in domestic capital goods expose households to idiosyncratic capital income risk in the style of Angeletos (2007): asset hk has a Brownian return with expected value r_{hk} and standard deviation φ_{hk}. Appendix E.1 provides the full set of model equations; we present the balanced growth equations below.

Balanced growth equations. In the model of Section 3, the balanced growth path was defined by equations (2)-(16). The balanced growth path in this section is defined by the same set of equations, with two modifications. First, the household budget constraint (2) is replaced by

$$p_h^C C_h = \sum_{f \in \mathcal{F}} w_{hf} L_{hf} + (r_0 - g) B_{h0} + \sum_{k \in \mathcal{K}} (r_{hk} - \tau_{hk}^K - g) B_{hk} + T_h, \tag{29}$$

which differs from (2) only by the inclusion of income from the risk-free bonds $(r_0 - g)B_{h0}$.

Second, in the model in this section, returns are not pinned down solely by the Euler equations. In Section 3, the long-run demand for assets is infinitely elastic, since the Euler equations (16) pin down the rates of return. The key change in this section is that we generalize the model to allow for an imperfectly elastic demand for assets. In the system of balanced growth equations, the Euler equations (16) are replaced by an asset demand system generated by the household problem.

In particular, there is a function $\mathcal{A}_h$ that maps expected returns, net of taxes, to desired savings. Aggregate assets on a balanced growth path satisfy

$$\frac{B_{h0} + \sum_{k\in\mathcal{K}} B_{hk}}{\sum_{f\in\mathcal{F}} w_{hf}L_{hf} + T_h} = \mathcal{A}_h(r_0, \boldsymbol{r}_h - \boldsymbol{\tau}_h^K), \tag{30}$$

where the left-hand side is the total value of household assets in country h relative to non-asset income. Furthermore, there is a function $\mathcal{A}_h^{risk}$ that determines total risk exposure:

$$\frac{\sum_k (r_{hk} - \tau_{hk}^K - r_0)B_{hk}}{\sum_{f\in\mathcal{F}} w_{hf}L_{hf} + T_h} = \mathcal{A}_h^{risk}(r_0, \boldsymbol{r}_h - \boldsymbol{\tau}_h^K), \tag{31}$$

where the left-hand side is excess return from risky assets relative to non-asset income. The functional forms for $\mathcal{A}_h$ and $\mathcal{A}_h^{risk}$ are in the appendix. Finally, there is a no-arbitrage condition linking the risk premia of different capital assets in the same country to their levels of idiosyncratic risk φ_{hk}:

$$\frac{(r_{hk} - \tau_{hk}^K) - r_0}{\varphi_{hk}} = \frac{(r_{hk'} - \tau_{hk'}^K) - r_0}{\varphi_{hk'}} \qquad k, k' \in \mathcal{K}. \tag{32}$$

This implies that Sharpe ratios of different risky assets are equalized in a country. In addition to these asset demand equations, there is a market clearing condition for the risk-free bond, which is in zero net supply:

$$0 = \sum_h B_{h0}. \tag{33}$$

Definition 3 (Balanced Growth Path with Endogenous Returns)**.** A balanced growth path with endogenous returns is a set of quantities, prices, and returns, such that (3)-(15) and (29)-(33) are satisfied.

Equivalence result. The following proposition extends the equivalence between balanced growth equilibria and distorted static economies in Proposition 2 to this setting.

Proposition 10 (Equivalence with Endogenous Returns)**.** *Consider a balanced growth path of an economy with returns r_{hk}, r_0, and foreign asset positions B_{h0}.*

1. *The balanced growth path is an equilibrium of the equivalent static economy in Proposition 2 with as-if capital wedges $\mu^K_{hk} = \frac{r_{hk}+\delta_k}{g+\delta_k}$ and cross-country transfers $\tilde{\mathcal{T}}_h = \mathcal{T}_h + (r_0 - g)B_{h0}$.*
2. *The returns r_0, r_{hk}, the net foreign asset positions B_{h0}, and the holdings of capital claims $B_{hk} = p^X_{hk}K_{hk}$ satisfy* (30)-(33).

Intuitively, the proposition shows that the balanced growth problem with endogenous returns can be solved in two steps. For given rates of return and foreign asset positions, the balanced growth path is an equilibrium of an equivalent static economy with capital wedges determined by the rates of return, and transfers given by the net foreign asset income. Given equilibrium prices and quantities from that equivalent static economy, those returns and net foreign asset positions are then pinned down by the asset market clearing equations (30)-(33).

The equivalence means that long-run comparative statics can still be characterized in terms of expenditure shares, wedges, and elasticities of substitution, just as in Proposition 3. The difference is that we have to add auxiliary equations that pin down changes in rates of return and net foreign asset positions from equations (30)-(33). The information required is the same as in Proposition 3, together with two new inputs: the initial net foreign asset positions and the elasticity of asset demand with respect to returns. The main conceptual difference relative to Proposition 3 is that if a shock changes the rates of return or net foreign asset positions, then this affects balanced growth variables in a way that is isomorphic to a change in wedges and transfers in the equivalent static economy.

To illustrate the effect of endogenous returns, the example below shows how comparative statics change in the neoclassical growth model when returns are endogenous.

Example 6 (Neoclassical Growth Model with Endogenous Returns)**.** We revisit Example 2, which studies the effect of productivity improvements in a standard one-sector neoclassical growth model with a CES aggregate production function, but we allow the rate of return on capital to be endogenous. We assume that there is no idiosyncratic capital income risk, which means that the return on physical capital is the same as on the risk-free bond $r_K = r_0$.

Proposition 10 implies that the effect of a productivity shock is given by the effect of the shock in the equivalent static economy, holding capital wedges μ^K fixed, plus the effect of the change in the capital wedges in the equivalent static economy from the change in returns. The full effect can be written as

$$\frac{d\log C}{d\log A_Y} = \frac{d\log C^{\text{exog. returns}}}{d\log A_Y} \underbrace{- \left[\frac{RK - pX}{pC}\right] \frac{\sigma_{KL}}{1-\alpha}}_{\frac{\partial \log C}{\partial \log \mu_K}} \underbrace{\frac{1}{r+\delta} \frac{dr}{d\log A_Y}}_{\frac{d\log \mu^K}{d\log A_Y}},$$

where $d\log C^{\text{exog. returns}}$ is the effect found in Example 2, which holds returns fixed, and the remainder is the effect of the change in the rate of return. The change in returns, dr, changes the as-if wedge on capital by $d\log \mu^K = \frac{dr}{r+\delta}$. As described in Proposition 5, the change in the capital wedge, $d\log \mu^K$, changes consumption in accordance with the deviation from the Golden Rule, captured by $(RK - p_X X)$, and the response in K to the wedge is $-\sigma_{KL}/(1-\alpha)$. Here, σ_{KL} is the elasticity of substitution between labor and capital in production and α is the capital share of GDP.

It remains to solve for the change in the rate of return dr. Asset market clearing — equations (15) and (30) — implies

$$\frac{pK}{wL} = \mathcal{A}(r)$$

where the left-hand side is the value of the capital stock relative to labor income, determined by firms' demand for capital services relative to labor, and the right-hand side is desired asset holdings, determined by households' willingness to save. The CES technology implies that the left-hand side is $\frac{pK}{wL} = \frac{\alpha}{1-\alpha}\left(\frac{p}{w}\right)^{1-\sigma_{KL}}(r+\delta)^{-\sigma_{KL}}$, where p is the price of the final good. In Appendix E.3, we show that for the special case with log utility ($\gamma = 1$), zero growth, and no idiosyncratic risk, the right-hand side is

$$\mathcal{A}(r) = \frac{r-\rho}{(r+\nu)(\nu+\rho-r)}, \qquad \rho < r < \rho + \nu,$$

where ρ is the discount rate and ν is the mortality rate. As $\nu \to 0$, the admissible interval $\rho < r < \rho + \nu$ collapses to $r = \rho$. Equivalently, asset supply becomes infinitely elastic around the representative-agent return.

The asset market clearing condition pins down the rate of return r. Differentiating the asset market clearing condition and rearranging yields[22]

$$\frac{dr}{d\log A_Y} = \frac{\frac{\sigma_{KL}-1}{1-\alpha}}{\frac{\mathcal{A}'(r)}{\mathcal{A}(r)} + \frac{\sigma_{KL}-\alpha}{1-\alpha}\frac{1}{r+\delta}}.$$

[22]We also use the fact that, by Shephard's lemma, $d\log p = -d\log A_Y + \alpha d\log(p(r+\delta)) + (1-\alpha)d\log w$. This expression can be rearranged to get $d\log p/w = 1/(1-\alpha)[-d\log A_Y + \alpha/(r+\delta)dr]$.

The productivity shock raises the rate of return r whenever $\sigma_{KL} > 1$, capturing that a positive productivity shock increases the value of capital relative to labor income if capital and labor are substitutable. The denominator is the sum of the elasticities of asset demand (from households) and capital demand (from firms). There is no response if, as in the neoclassical growth model, asset demand is infinitely elastic, $\mathcal{A}'(r) = \infty$.

Putting these equations together gives the full comparative static. We note that in general, the adjustment in returns tends to dampen the power of reallocation: if $\sigma_{KL} > 1$, returns generally rise, which dampens the capital response to increases in TFP A_Y, and vice versa when $\sigma_{KL} < 1$. At the Golden Rule, the response in returns is irrelevant for the change in long-run consumption.

8 Quantitative Illustrations

We use a calibrated version of our model to explore the quantitative relevance of our analytical results for both changes in productivities and changes in wedges. We use the model to show the importance of correctly accounting for capital accumulation and the Golden Rule wedge. For changes in productivity, we show that the simple-to-compute cost-based Domar weights $\tilde{\lambda}_i$ and $\tilde{\lambda}_i^{\text{soe}}$ provide an excellent approximation of the long-run effects on global and country-level consumption. For wedges, we focus on tariffs, and show that the capital adjustment mechanism is much stronger than traditional terms-of-trade effects for long-run outcomes.

8.1 Environment

Our baseline analysis uses the model in Section 3. We also consider an extension with open capital markets and endogenous returns as in Section 7.

Recall from Section 3 that there is a set $\mathcal{H}$ of countries, each with one consumption good, as well as a set $\mathcal{N}$ of intermediate goods and $\mathcal{K}$ investment goods, which are used to accumulate associated capital stocks. We assume that each intermediate industry i uses a unique associated capital good $k(i)$, so that $|\mathcal{N}| = |\mathcal{K}|$. Production is given by

$$Y_h^C = A_h^C \prod_{i \in \mathcal{N}} \left(Y_{h,i}^C \right)^{\tilde{\Omega}_{h,i}^C},$$

$$Y_{hi}^Y = A_{hi}^Y \prod_{i' \in \mathcal{N}} \left(Y_{hi,i'}^Y \right)^{\tilde{\Omega}_{hi,i'}^Y} K_{hi,hk(i)}^{\tilde{\Omega}_{hi,hk(i)}^Y} \prod_{f \in \mathcal{F}} L_{hi,hf}^{\tilde{\Omega}_{hi,hf}^Y}$$

$$X_{hk} = A^X_{hk} \prod_{i' \in \mathcal{N}} \left(Y^X_{hk,i'} \right)^{\tilde{\Omega}^X_{hk,i'}},$$

where $Y^C_{h,i'}$, $Y^Y_{hi,i'}$ and $Y^X_{hk,i'}$ are Armington aggregates that feature a constant elasticity of substitution across inputs from different origins:

$$Y^u_{a,i'} = \left(\sum_{h'} \omega_{a,h'i'} (Y^u_{a,h'i'})^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} \qquad u \in \{C, Y, X\}, \quad a \in \{h, hi, hk\},$$

with θ being the elasticity across varieties from different countries. For example, machinery in the US has a Cobb-Douglas production function across labor, machinery-specific capital, and input bundles from mining, construction, and so forth. The mining bundle used by machinery in the US is a CES aggregate of mining output from different countries. All share parameters are specific to the destination industry and country, which means that we can exactly match global input-output and investment flow tables with countries and industries.

8.2 Solution method and calibration

We solve the model in terms of deviations from an initial balanced growth path. Proposition 3 shows that first-order comparative statics are pinned down by the initial input cost matrix $\tilde{\Omega}$, consumption shares by country Φ_h, initial wedges $\boldsymbol{\mu}$, and the Armington elasticity θ.[23] These objects are also sufficient for determining nonlinear effects. To obtain nonlinear effects, we iterate the first-order comparative statics, update the variables, and cumulate changes; see Appendix F.1 and Baqaee and Farhi (2024) for more details on this approach to obtaining nonlinear results.

We calibrate the model to a balanced growth path with nine regions (the US, EU, Japan, China, UK, India, Canada, Mexico, and a rest-of-world composite). The calibration uses the World Input-Output Database (WIOD) and its associated Socio-Economic Accounts (Timmer et al., 2015), augmented with the global investment flow matrix of Ding (2022). We provide a brief summary of the calibration here and leave the explicit details in Appendix F.2. Since we only have investment-flow data for the year 1997, we use the earlier part of the World Input-Output Database, averaging across the years 1995 to 2009.

The household consumption share Φ_h is h's share of world final consumption.[24] Each

[23] In particular, Proposition 3 states that comparative statics are pinned down by the input cost matrix $\tilde{\Omega}$, expenditure shares Φ, initial wedges $\boldsymbol{\mu}$, and an array of Allen elasticities of substitution $\boldsymbol{\sigma}$. The last one is pinned down by the input cost matrix and the CES elasticity θ as shown in Appendix F.2.

[24] Since capital markets are closed in our baseline, this implicitly means we match trade deficits using ex-

input cost share $\tilde{\Omega}^{C}_{h,h'i}$ is given by the corresponding share of h's consumer spending that goes to the country-industry pair (h', i). We calibrate the production cost shares $\tilde{\Omega}^{Y}$ using the WIOD, setting capital rental costs equal to capital compensation in the Socio-Economic Accounts. We calibrate the cost shares $\tilde{\Omega}^{X}$ for the investment goods using the investment flow matrix constructed by Ding (2022).

We calibrate the Golden Rule wedges, μ^{K}_{hk}, to the ratio of capital income to investment spending for the capital good associated with each industry and country. We assume the dynamic economy is perfectly competitive and that there are zero initial tariffs, so all other wedges $\mu^{Y} = 1$.

Table 3: Capital wedges on the balanced growth path for the four largest economies

Parameter	**Description**	**USA**	**CHN**	**EU**	**JPN**
$\bar{\mu}^{K}_{h}$	Harmonic average wedge on capital	2.363	1.796	2.418	2.141

Notes: The table reports the harmonic average capital wedge across industries within each selected economy on the calibrated balanced growth path.

As emphasized throughout the paper, the Golden Rule wedges μ^{K}_{hk} are central to long-run comparative statics. Table 3 reports average Golden Rule wedges for a number of large economies. These are typically around two, which is in line with capital income being roughly twice as high as investment expenditures. Since the wedge is much greater than one, the initial equilibrium is far from the Golden Rule, creating significant scope for reallocation effects and differences between sales and cost weights.

8.3 Productivity Changes

We consider the long-run consumption effects of productivity shocks, and show that at both the global and country levels, simple-to-compute cost weights, $\tilde{\lambda}$ and $\tilde{\lambda}^{\text{soe}}$, provide an excellent approximation to the comparative static from the full model.

Global effects. Our first experiment considers the global effect of changes in industry productivities: we raise the productivity of industry $i \in \mathcal{N}$ in all countries simultaneously and study the effect on global consumption. Panel A in Table 4 reports the effect for selected industries. The first column reports the comparative static from the full model solution. The second column reports the simple-to-compute cost-based Domar weights $\sum_{h \in \mathcal{H}} \tilde{\lambda}_{Y_{hi}}$ from Section 5, which are almost identical to the full model outcomes.

ogenous transfers $\mathcal{T}_h$, which are held fixed as a share of nominal world consumption in doing comparative statics (Dekle et al., 2008).

Table 4: Long-run consumption effect of industry TFP changes for selected industries.

Panel A: Global

Sector	**Full effect** $\sum_h \frac{\partial \log C}{\partial \log A^Y_{hi}}$	**Cost-based weight** $\sum_h \tilde{\lambda}_{Y_{hi}}$	**Sales / Aggregate labor share** $\frac{\sum_h \lambda_{Y_{hi}}}{\sum_{h,f} \lambda_{L_{hf}}}$
Agriculture	0.095	0.095	0.121
Basic and fabricated metals	0.187	0.184	0.119
Machinery	0.126	0.125	0.068
Construction	0.385	0.379	0.197
Hotels and restaurants	0.074	0.074	0.096
Health	0.105	0.105	0.152

Panel B: India-only

Sector	**Full effect** $\frac{\partial \log C_h}{\partial \log A^Y_{hi}}$	**Cost-based weight** $\tilde{\lambda}^{\text{soe}}_{Y_{hi}}$	**Sales / Aggregate labor share** $\frac{\lambda_{Y_{hi}}}{\sum_f \lambda_{L_{hf}}}$
Agriculture	0.434	0.403	0.698
Basic and fabricated metals	0.421	0.400	0.292
Machinery	0.200	0.196	0.143
Construction	0.840	0.801	0.403
Hotels and restaurants	0.062	0.055	0.092
Health	0.042	0.042	0.083

Notes: Panel A raises productivity in industry i in every country simultaneously and reports the first-order effect on global consumption. Panel B raises productivity in industry i in India alone and reports the first-order effect on Indian consumption. The cost-based benchmark is the global cost-based Domar weight in Panel A and the small-open-economy cost-based Domar weight in Panel B. The sales/labor-share benchmark divides the relevant revenue Domar weight by aggregate labor income.

As a point of comparison, we also report the classic capital amplification formula, which scales changes in aggregate TFP by the inverse of the labor share. In our framework, this corresponds to industry sales divided by aggregate labor income.[25] The third column shows that this measure can deviate considerably from the full effect and from the cost weight. The classic formula overstates the long-run importance of downstream industries such as health care and hotels and restaurants, and understates the importance of investment-related industries such as construction and machinery. The error stems from the classic formula ignoring industries' positions in the supply chain of investment goods.

[25]Specifically, by Hulten's theorem, the change in aggregate TFP due to a change in industry productivity is sales divided by GDP. Dividing by the labor share yields sales over aggregate labor income.

Country-level effects. Our second experiment considers the country-level effect of changes in productivity. Panel B in Table 4 reports the results, using India as an example. The first column reports the long-run response of India's consumption to a permanent increase in the productivity of each industry in India. The second column shows that these responses are well approximated by each industry's simple-to-compute small open economy cost weight $\tilde{\lambda}^{soe}_{Y_{hi}}$. The third column shows the standard development accounting formula, which divides TFP effects by the labor share. Again, the standard formula does worse, understating the effects of industries that are upstream of investment, and overstating the effects of those that are not. The standard formula also does not account for terms-of-trade effects, which reduce the cost weight of exporting industries.

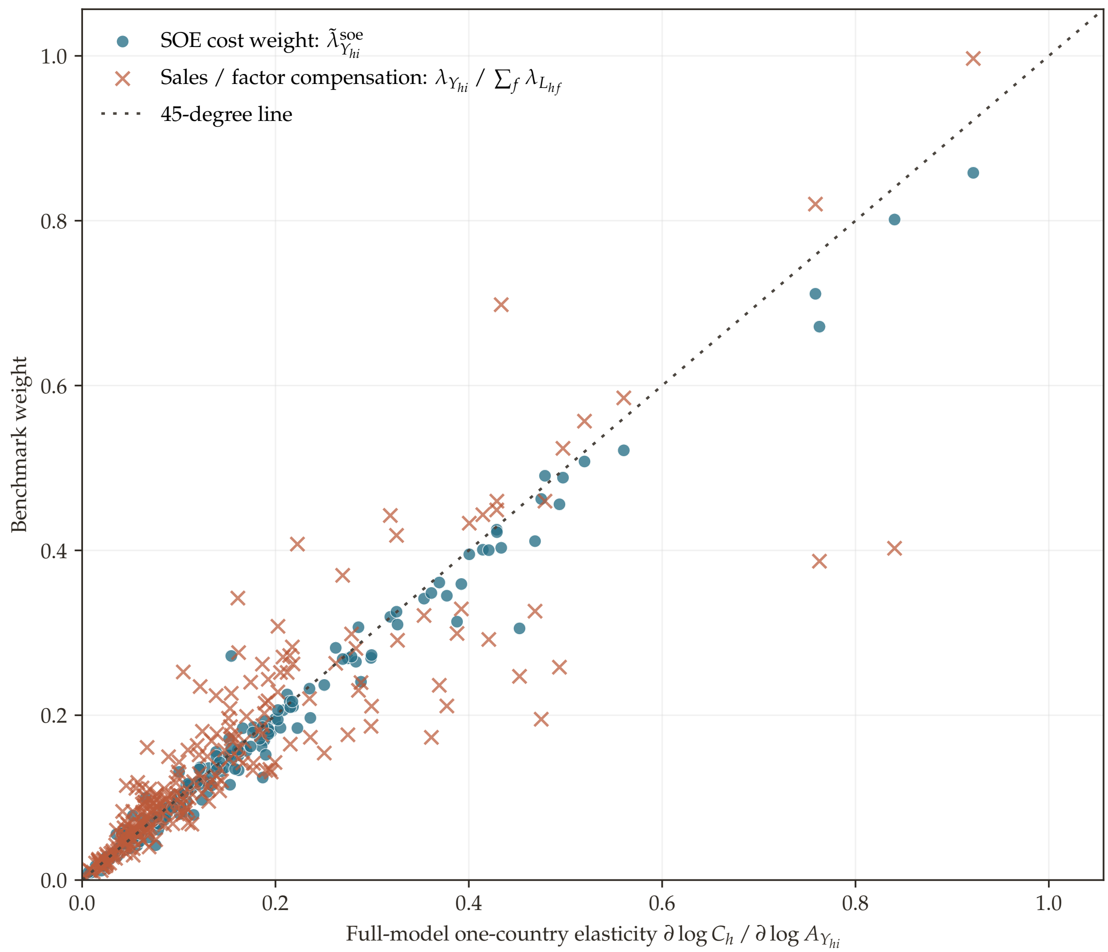

Figure 4: Comparison of the small open economy weight $\tilde{\lambda}^{soe}_{Y_{hi}}$ to full model results

To assess how well $\tilde{\lambda}^{soe}_{Y_{hi}}$ approximates country-level results more systematically, we conduct the same experiment for all countries and industries, comparing $\tilde{\lambda}^{soe}_{Y_{hi}}$ to the full model comparative static. Figure 4 shows that the cost weight is an excellent approximation. For completeness, the figure also includes the standard capital amplification formula, showing that it is a worse approximation.[26]

Taken together, the simple measures $\tilde{\lambda}$ and $\tilde{\lambda}^{\text{soe}}$ provide an excellent approximation to the full quantitative results. This close fit reflects that, in the benchmark calibration, the small open economy approximation is good for country-level effects, and productivity shocks in general induce only limited reallocation.

8.4 Effect of Tariffs

In this section, we consider the long-run consumption effects of distortions. As Proposition 5 establishes, if non-capital wedges are small, $\mu^Y \approx 1$, then the only thing that determines the long-run response of consumption to a wedge change is, to first order, the change in the capital stock. That is, what matters are the shifts and elasticities of capital supply and demand.

In this section, we use import tariffs as an example to demonstrate this. As in Example 3, we consider a uniform increase in tariffs across countries, starting from no initial tariffs. We show that at both the global and country levels, the key determinant of long-run consumption is the interaction between changes in capital stocks and initial Golden Rule wedges. Just as in Example 3, the elasticities that matter are the ones that determine capital supply and demand, and not the ones in the market for traded goods where the tariffs themselves appear (e.g., the Armington trade elasticity).[27]

Global effects. The first column in Table 5 reports the effect on global consumption from the uniform tariff increase under different parameterizations of the model. The first-order effect is scaled to capture a 20 percent tariff.

In the benchmark calibration, a 20 percent tariff lowers long-run global consumption by a bit more than 3 percent. The mechanism is the one highlighted in Example 3: tariffs raise investment prices, leading to depressed capital accumulation which interacts with the economy's deviation from the Golden Rule. The alternative parameterizations illustrate the central role of this channel.

[26] There are industries where the amplification — the cost weight divided by the sales weight — is very different between the global and country-level comparisons. For example, the ratio of $\tilde{\lambda}^{\text{soe}}/\lambda^{\text{soe}}$ is smaller for Canadian electrical equipment than for world electrical equipment. This is because much of Canadian electrical equipment is exported, so it is more important for the global capital stock than it is for the domestic capital stock.

[27] A number of papers have studied trade policy in dynamic models with endogenous capital accumulation; see, for example, Alvarez (2017) and Ravikumar et al. (2019). The novel contribution here is to emphasize the central role of the interaction between changes in capital stocks and the economy's initial distance from the Golden Rule.

Table 5: Global consumption effects from uniform 20 percent tariffs

Scenario	Description	First-order	Nonlinear
Benchmark	Baseline	-0.035	-0.033
Static	No capital adjustment (static calibration)	0.000	-0.007
$\delta \to \infty$	Set $\mu^K_{hk} = 1$ for all h,k	0.000	-0.016
$\sigma_{KL} = 0.8$	Lower capital-labor substitutability	-0.024	-0.026
$\sigma_{KL} = 1.2$	Higher capital-labor substitutability	-0.045	-0.040
OLG	Endogenous returns and open capital markets	-0.026	-0.028
$\theta = 1$	Unit Armington elasticity	-0.032	-0.032

Notes: Each row reports the global consumption response to a uniform 20 percent tariff. First-order responses use the linearized model; nonlinear responses iterate the comparative statics for 20 steps. Fixed-rate cases impose closed international capital markets, while OLG uses open markets and endogenous returns.

First, we note that the first-order effect is precisely zero when capital is held fixed or when the economy is calibrated to be at the Golden Rule. This follows from Proposition 5, since the negative consumption effect depends on the interaction of changes in capital stocks and capital wedges. Furthermore, since the elasticity of substitution between capital and labor determines how much capital stocks respond to higher investment prices, the magnitude of the consumption effect is increasing in this elasticity, with smaller declines when $\sigma_{KL} = 0.8$, and larger declines when $\sigma_{KL} = 1.2$.[28] Similarly, when rates of return are endogenous and asset demand is not perfectly elastic, the effect of tariffs is dampened. Intuitively, the tariff lowers investment demand, which lowers the rate of return, and the reduction in the rate of return cushions the decline in investment.[29] In contrast to the supply and demand elasticities in the capital market, the trade elasticities are essentially irrelevant. This is because it is trade shares rather than trade elasticities that are the central determinants of first-order investment price responses.

The fact that consumption responses are precisely zero without the capital effect reflects that we study first-order effects. However, even for larger changes, the capital effect remains central. The second column shows the nonlinear effect (computed by iterating on Proposition 3 as described in Appendix F.1). Once we consider such large changes, consumption decreases even in the case with fixed capital, but the effect is still much smaller than the baseline result.

[28]Here, σ_{KL} is the microeconomic elasticity of substitution between capital and labor in individual production functions. So, for each industry-country pair we let the value-added component of production be a CES aggregator with elasticity σ_{KL} instead of a Cobb-Douglas aggregator.

[29]See Appendix F.3 for a description of how the model with endogenous returns is calibrated.

Country-level effects. Capital adjustment is also central for country-specific effects. To quantify their importance, we rely on the following generalization of Proposition 5 to country-level responses to tariffs (see Appendix F.5 for the derivation):

$$d\log C_h = \underbrace{\sum_{k\in\mathcal{K}} \frac{R_{hk}K_{hk} - p^X_{hk}X_{hk}}{p^C_h C_h} d\log K_{hk}}_{\text{Capital adjustment}} + \underbrace{\sum_{h'\in\mathcal{H},i\in\mathcal{N}} \frac{NX_{h,h'i}}{p^C_h C_h} d\log p^Y_{h'i}}_{\text{Terms of trade}}, \tag{34}$$

where $NX_{h,h'i}$ is defined to be net exports of good $h'i$ for country h.[30] In this decomposition, the first term is the capital adjustment effect, as in Proposition 5, and the second term is the terms-of-trade effect. The second term is positive if export prices rise relative to import prices. Table 6 presents this decomposition using our baseline model. We see that the capital effect is the main driver of country-level effects, with only a minor role for the terms-of-trade effects often emphasized in the trade literature.

Table 6: Decomposition of consumption changes using equation (34) for selected regions

Country	**Consumption** $d\log C_h$	**Capital adjustment**	**Terms of trade**
United States	-0.021	-0.021	0.000
Canada	-0.088	-0.079	-0.009
China	-0.043	-0.034	-0.009
Mexico	-0.204	-0.211	0.007
European Union	-0.023	-0.022	-0.000
Global	-0.035	-0.035	0.000

Notes: The table decomposes the first-order country consumption response to a uniform 20 percent tariff under fixed interest rates. The global terms-of-trade contribution sums to zero.

Table 6 underscores the importance of accounting for capital accumulation and the Golden Rule wedge when considering how changes in wedges, like tariffs, affect long-run consumption.

[30] The decomposition in (34) assumes trade balance. If trade is imbalanced, there is technically another term in the decomposition capturing changes in the transfers. In practice, this term is very small because trade imbalances are small in the data, so we ignore them here for expositional simplicity.

9 Conclusion

We provide a framework for analyzing long-run comparative statics of dynamic economies by representing them as distorted static economies. This representation allows us to use tools from the static literature to study these models. We characterize comparative statics in terms of expenditure shares, elasticities of substitution, and initial wedges. The recurring theme is that long-run consumption responses can be understood through second-best principles, even in dynamically efficient economies.

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Online Appendix

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A Appendix to Section 3: Model

A.1 Dynamic Setup

The dynamic household problem is presented in the main text. In the time-dependent economy, the production functions (3)-(5) are the same but with output and inputs being time-dependent. The same is true for the pricing equations (7) and resource constraints (10)-(14). The evolution equation of the capital stocks is given by

$$\dot{K}_{hk} = X_{hk}(t) - \delta_k K_{hk}(t) \qquad h \in \mathcal{H}, \quad k \in \mathcal{K}. \tag{35}$$

The user cost formula (8) and asset market clearing conditions (15) can be derived from the asset market setup. For each capital good hk, there is an intermediary that invests in capital and rents it out to firms at a rental rate $R_{hk}(t)$. It maximizes dividends $R_{hk}(t)K_{hk}(t) - p^X_{hk}(t)X_{hk}(t)$ discounted by the rate of return $r_{hk}(t)$. Given the current capital stock, let $V_{hk}(t,K)$ denote the value of an intermediary of type hk at date t with capital stock K. Its recursive problem is

$$r_{hk}(t)V_{hk}(t,K) = \frac{\partial V_{hk}(t,K)}{\partial t} + \max_{X \geq 0}\left\{R_{hk}(t)K - p^X_{hk}(t)X + \frac{\partial V_{hk}(t,K)}{\partial K}\left[X - \delta_k K\right]\right\}. \tag{36}$$

The problem is homogeneous of degree one in the capital stock, so $V_{hk}(t,K) = q_{hk}(t)K$.[31] Substituting and collecting terms,

$$\left[r_{hk}(t) + \delta_k\right] q_{hk}(t)K = \dot{q}_{hk}(t)K + R_{hk}(t)K + \max_{X \geq 0}\left\{\left[q_{hk}(t) - p^X_{hk}(t)\right] X\right\}. \tag{37}$$

Investment enters linearly, so a finite value requires $q_{hk}(t) \leq p^X_{hk}(t)$, and $X_{hk}(t) > 0$ requires

$$q_{hk}(t) = p^X_{hk}(t). \tag{38}$$

That is, the intermediary is worth the replacement value of its capital stock. The maximum in (37) is then zero, and differentiating (38) and substituting gives the user cost of capital,

$$R_{hk}(t) = p^X_{hk}(t)\left[r_{hk}(t) + \delta_k - \frac{\dot{p}^X_{hk}(t)}{p^X_{hk}(t)}\right]. \tag{39}$$

[31] Payoffs and the law of motion are linear in (K, X), and the constraint $X \geq 0$ is invariant to scaling X by a positive multiple. Hence, for any $\alpha > 0$, scaling the initial stock by α scales the set of feasible paths and their payoffs by α. This means that $V_{hk}(t,K) = KV_{hk}(t,1)$, where $V_{hk}(t,1)$ is the value associated with a unit capital stock.

Since households own the intermediary, their asset holdings satisfy

$$B_{hk}(t) = V_{hk}\big(t, K_{hk}(t)\big) = p^X_{hk}(t)K_{hk}(t). \tag{40}$$

On a balanced growth path, prices are constant and $\dot{K}_{hk} = gK_{hk}$, so

$$X_{hk} = (g + \delta_k)K_{hk} > 0$$

for every active capital good and (39) applies. Therefore,

$$R_{hk} = p^X_{hk}(r_{hk} + \delta_k), \tag{41}$$

$$B_{hk} = p^X_{hk}K_{hk}, \tag{42}$$

which are the user-cost and asset-market-clearing equations used in the main text.

B Appendix to Section 4: Equivalence Result and Comparative Statics

B.1 Elements of the input-output matrix

The nonzero elements of $\tilde{\Omega}$ are:

Consumption:

$$[\tilde{\Omega}_{H,HN}]_{h,h'i'} = \frac{p^Y_{h'i'}Y^C_{h,h'i'}}{\sum\limits_{\substack{h''\in\mathcal{H}\\ i''\in\mathcal{N}}} p^Y_{h''i''}Y^C_{h,h''i''}}, \qquad h,h' \in \mathcal{H},\ i' \in \mathcal{N}, \tag{43}$$

Standard goods:

$$[\tilde{\Omega}_{HN,HN}]_{hi,h'i'} = \frac{p^Y_{h'i'}Y_{hi,h'i'}}{p^Y_{hi}Y_{hi}/\mu^Y_{hi}}, \qquad h,h' \in \mathcal{H},\ i,i' \in \mathcal{N}, \tag{44}$$

$$[\tilde{\Omega}_{HN,HK}]_{hi,hk} = \frac{R_{hk}K_{hi,hk}}{p^Y_{hi}Y_{hi}/\mu^Y_{hi}}, \qquad h \in \mathcal{H},\ i \in \mathcal{N},\ k \in \mathcal{K}, \tag{45}$$

$$[\tilde{\Omega}_{HN,HF}]_{hi,hf} = \frac{w_{hf}L_{hi,hf}}{p^Y_{hi}Y_{hi}/\mu^Y_{hi}}, \qquad h \in \mathcal{H},\ i \in \mathcal{N},\ f \in \mathcal{F}, \tag{46}$$

Investment goods:

$$[\tilde{\Omega}_{HX,HN}]_{hk,h'i'} = \frac{p^Y_{h'i'} Y^X_{hk,h'i'}}{p^X_{hk} X_{hk}}, \qquad h, h' \in \mathcal{H},\ k \in \mathcal{K},\ i' \in \mathcal{N}, \tag{47}$$

where we use that $p^Y_{hi} Y_{hi} / \mu^Y_{hi}$ equals the total costs of industry (h, i).

B.2 Extensions of the Equivalence Result

B.2.1 Non-Homothetic Preferences.

Our equivalence results can be extended to economies with non-homothetic preferences. We focus on steady states in economies without growth. With non-homothetic preferences, growth in household expenditure generally changes the composition of demand, so the consumption allocation need not be stationary after detrending. Setting growth to zero allows us to extend the equivalence result without imposing additional restrictions on income effects.

To model non-homotheticities, we remove the consumption-good industries and allow households to consume intermediate goods directly. Let $\mathbf{C}_h(t) = \{C_{h,h'i}(t)\}_{h' \in \mathcal{H}, i \in \mathcal{N}}$ denote the consumption vector of household h, $u_h(\mathbf{C}_h)$ denote its period utility, and $v_h(E, \boldsymbol{p})$ its indirect utility function, where $\boldsymbol{p}^Y = \{p^Y_{hi}\}_{h \in \mathcal{H}, i \in \mathcal{N}}$ are the prices of intermediate goods. The dynamic household problem is

$$\max_{\{E_h(t), B_{hk}(t)\}} \int_0^\infty e^{-\rho_h t} v_h\big(E_h(t), \mathbf{p}(t)\big)\, dt \tag{48}$$

subject to

$$E_h(t) + \sum_{k \in \mathcal{K}} \dot{B}_{hk}(t) \leq \sum_{f \in \mathcal{F}} w_{hf}(t) L_{hf} + \sum_{k \in \mathcal{K}} \big(r_{hk}(t) - \tau^K_{hk}\big) B_{hk}(t) + T_h(t). \tag{49}$$

In a steady state with positive investment, $\frac{\partial v_h}{\partial E}$ is a constant, which means that interest rates are still given by the Euler relationship (16). We obtain the following equivalence result.

Proposition 11 (Equivalence with non-homothetic preferences)**.** *Given parameters A, μ^Y, and $\boldsymbol{\tau}^K$, the steady-state prices and quantities of the dynamic economy are the same as the prices and quantities of a static economy. In this static economy, (i) the wedges μ^Y and the production functions for intermediate and investment goods are the same as in the dynamic economy, and each household has the same non-homothetic preferences over intermediate goods; (ii) capital stocks K_{hk} are replaced by capital services that are produced instantaneously from investment goods according* *to the production function*

$$K_{hk} = A^K_{hk} X_{hk}, \qquad A^K_{hk} = \frac{1}{\delta_k};$$

(iii) capital services are sold at a markup

$$\mu^K_{hk} = \frac{r_{hk} + \delta_k}{\delta_k},$$

with markup revenue from capital good k in country h distributed to the household in that country, and

$$r_{hk} = \rho_h + \tau^K_{hk},$$

and transfers equal those in the dynamic economy.

Proof. The equilibrium of the static economy in the proposition is characterized by the production functions (4)-(5), the production function for capital services $K_{hk} = X_{hk}/\delta_k$, cost-minimization given prices and production functions, prices given by (7), $R_{hk} = \mu^K_{hk}\mathcal{C}^K_{hk} = \mu^K_{hk}\delta_k\mathcal{C}^X_{hk}$, and satisfaction of the resource constraints. Consumption quantities in country h maximize u_h subject to prices and total expenditure E_h, which satisfies

$$E_h = \sum_{f\in\mathcal{F}} w_{hf}L_{hf} + \sum_{i\in\mathcal{N}}(1 - 1/\mu^Y_{hi})p^Y_{hi}Y_{hi} + \sum_{k\in\mathcal{K}}(1 - 1/\mu^K_{hk})R_{hk}K_{hk} + \mathcal{T}_h.$$

Now suppose that we have a steady state in the dynamic economy. The production functions, the steady-state capital equations, the price equations (7), and the resource constraints coincide with those of the static economy. Since $R_{hk} = (r_{hk} + \delta_k)p^X_{hk}$, we also have $R_{hk} = \mu^K_{hk}\delta_k\mathcal{C}^X_{hk}$, since $p^X_{hk} = \mathcal{C}^X_{hk}$. Household optimality means that consumption quantities maximize u_h given E_h and prices. The household budget constraint in the dynamic economy satisfies

$$\begin{aligned}
E_h &= \sum_{f\in\mathcal{F}} w_{hf}L_{hf} + \sum_{k\in\mathcal{K}}(r_{hk} - \tau^K_{hk})B_{hk} + T_h \\
&= \sum_{f\in\mathcal{F}} w_{hf}L_{hf} + \sum_{k\in\mathcal{K}}(r_{hk} - \tau^K_{hk})p^X_{hk}K_{hk} + \left(\sum_{i\in\mathcal{N}}(1 - 1/\mu^Y_{hi})p_{hi}Y_{hi} + \sum_{k\in\mathcal{K}}\tau^K_{hk}p^X_{hk}K_{hk} + \mathcal{T}_h\right) \\
&= \sum_{f\in\mathcal{F}} w_{hf}L_{hf} + \sum_{i\in\mathcal{N}}(1 - 1/\mu^Y_{hi})p_{hi}Y_{hi} + \sum_{k\in\mathcal{K}} r_{hk}p^X_{hk}K_{hk} + \mathcal{T}_h \\
&= \sum_{f\in\mathcal{F}} w_{hf}L_{hf} + \sum_{i\in\mathcal{N}}(1 - 1/\mu^Y_{hi})p_{hi}Y_{hi} + \sum_{k\in\mathcal{K}} \frac{r_{hk}}{r_{hk} + \delta_k}(r_{hk} + \delta_k)p^X_{hk}K_{hk} + \mathcal{T}_h
\end{aligned}$$

$$= \sum_{f\in\mathcal{F}} w_{hf}L_{hf} + \sum_{i\in\mathcal{N}} (1-1/\mu^{Y}_{hi})p_{hi}Y_{hi} + \sum_{k\in\mathcal{K}} \underbrace{\left[1-\frac{\delta_k}{r_{hk}+\delta_k}\right]}_{1-1/\mu^{K}_{hk}} R_{hk}K_{hk} + \mathcal{T}_h,$$

where we substitute in the income from wedges and rebated capital taxes on the second line. □

B.2.2 Firm entry models.

To illustrate how the results extend to non-physical capital, we analyze a model in the style of Hopenhayn (1992) in which the stock of firms operates as a capital good and entry costs operate as investment. There is a single country with labor endowment L, no growth ($g = 0$), and a single output good, with its price normalized to one. A continuum of perfectly competitive firms with productivity z produce output $y = z\ell^{\eta}$ from ℓ workers, with $\eta \in (0,1)$. To start a firm, entrepreneurs hire $1/z_e$ workers, where z_e is the productivity of the entry technology. Firms exit at rate δ. The household has the preferences of Section 3, so that the steady-state interest rate is $r = \rho$.

Firm behavior and aggregation. Each incumbent chooses employment to maximize variable profits $z\ell^{\eta} - w\ell$, which yields $w = \eta z\ell^{\eta-1}$ and per-firm profits $\pi = (1-\eta)z\ell^{\eta}$. With a measure M of identical incumbents and production labor $L_Y = M\ell$, aggregate output is

$$Y = zM^{1-\eta}L_Y^{\eta} \equiv zF[M, L_Y],$$

where F has constant returns to scale in (M, L_Y). The firm-level conditions imply

$$w = zF_{L_Y}, \qquad \pi = zF_M,$$

that is, the production structure behaves as if a representative firm hires labor at wage w and rents the stock of firms at rental rate π.

Entry and the stock of firms. The value of an incumbent is $v = \int_0^{\infty} e^{-(r+\delta)t}\pi\, dt = \pi/(r+\delta)$, and free entry requires $v = w/z_e$. Writing $X = z_eL_e$ for the flow of entrants created by L_e entry workers, the measure of firms evolves as $\dot{M} = X - \delta M$. We write B for the value of consumer claims on firms and $p^X = w/z_e$ for the marginal cost of creating a firm. Free entry implies that the value of all firms satisfies $B = p^X M$. Furthermore, profits net of entry costs are

$$M(\pi - \delta w/z_e) = rp^X M = rB.$$

Steady-state equations. Steady states satisfy

$$
\begin{aligned}
&C = wL + rB && \text{(budget constraint)}\\
&Y = zF[M, L_Y], && \text{(production)}\\
&X = z_e L_e, \qquad M = X/\delta, && \text{(entry and firm accumulation)}\\
&w = zF_{L_Y}, \qquad \pi = (r+\delta)\, p^X, && \text{(pricing and free entry)}\\
&C = Y, \qquad L_Y + L_e = L, && \text{(resource constraints)}\\
&B = p^X M && \text{(asset market clearing)}
\end{aligned}
$$

These are the balanced-growth equations of Section 3 with one country, $g = 0$, and the following relabeling: the measure of firms M is a capital stock with depreciation rate δ; the flow of entrants X is the associated investment good, produced by an industry with productivity z_e; firm values B are the corresponding asset; and free entry combined with the value of an incumbent is the user-cost equation (8).[32] Proposition 2 therefore applies directly.

Proposition 12 (Equivalence with firm entry and exit)**.** *Steady-state prices and quantities of the economy above are the same as the prices and quantities of a static economy in which (i) the production function for output is $Y = zF[M, L_Y]$; (ii) firm services M are produced from entry labor according to $M = (z_e/\delta)L_e$; and (iii) firm services are sold at a markup*

$$\mu^M = \frac{r+\delta}{\delta},$$

with markup revenue distributed to the household.[33]

The static economy is a static entry model with entry cost δ/z_e units of labor per firm, in which profits are taxed at rate $\tau = r/(r+\delta)$ and tax revenue is rebated to the household. One upshot is that for a static firm entry model to have the same long-run comparative statics as a dynamic model, a profit tax needs to be introduced.

[32] The only formal difference is that the entry technology uses labor, while investment goods in Section 3 use intermediate inputs. In practice, the proof of Proposition 2 does not restrict the inputs of the investment technology. Alternatively, entry labor can be routed through an intermediate industry supplying the investment sector, following the convention.

[33] With labor endowments growing at rate g, the same equations hold with $X = (g+\delta)M$ and $\mu^M = (r+\delta)/(g+\delta)$, since F has constant returns to scale.

B.2.3 Endogenous Growth.

The dynamic-static equivalence is also present in models where the growth rate is endogenous. We illustrate with a one-sector economy in the style of Romer (1986), in which there is an externality from the aggregate capital stock that raises labor productivity. There is one country, one labor endowment L, one capital good, and a markup wedge μ on the final good. Output is produced with the CES technology

$$Y(t) = A\left[\alpha K(t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)\big(\bar{K}(t)L\big)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{50}$$

where $\bar{K}(t)$ is the aggregate capital stock, which firms take as given, and σ is the elasticity of substitution between capital and labor. In equilibrium, $\bar{K}(t) = K(t)$. Aggregate capital enters as labor-augmenting technical change, so that private capital and effective labor grow at the same rate as capital accumulates. Output can be consumed or invested, $C(t) + X(t) = Y(t)$, capital is accumulated according to $\dot{K}(t) = X(t) - \delta K(t)$, and the household has the preferences of Section 3, with discount rate ρ and intertemporal elasticity of substitution γ.

We consider balanced growth paths on which $K(t) = \bar{K}(t) = e^{gt}K_0$ and all quantities grow at rate g, with the level of the path indexed by the initial capital stock K_0. The private marginal product of capital evaluated at the symmetric allocation $K = \bar{K}$,

$$\text{MPK} = A\alpha\left[\alpha + (1-\alpha)L^{\frac{\sigma-1}{\sigma}}\right]^{\frac{1}{\sigma-1}}, \tag{51}$$

is independent of the capital stock. Capital accumulation therefore does not run into diminishing returns, and growth is sustained endogenously. The growth rate follows from combining (51) with the household Euler equation $r = \rho + g/\gamma$ and the no-arbitrage condition $\mu(r+\delta) = \text{MPK}$.[34]

Proposition 13 (Equivalence with endogenous growth)**.** *Suppose the economy is on a balanced growth path with $K(t) = \bar{K}(t) = e^{gt}K_0$. Then balanced growth path prices and detrended quantities are the same as the prices and quantities of a static economy. In this static economy, (i) output is produced according to* (50)*, with the externality term fixed at $\bar{K} = K_0$; (ii) there are capital services produced instantaneously from investment goods according to $K = A^K X$, with $A^K = 1/(g+\delta)$; (iii) capital services are sold at a markup $\mu^K = (r+\delta)/(g+\delta)$, with markup*

[34] We assume parameters are such that the implied growth rate satisfies $g \geq 0$ and $\rho > (1-1/\gamma)g$; the latter guarantees $r > g$ and finite lifetime utility.

revenue distributed to the household and $r = \rho + g/\gamma$*; and (iv) the growth rate is*

$$g = \gamma \left(\frac{\text{MPK}}{\mu} - \delta - \rho \right), \tag{52}$$

where $\text{MPK} = \tilde{\Omega}_{Y,K} \times Y/K$ *and* $\tilde{\Omega}_{Y,K}$ *is the cost share of capital.*

Proof. Parts (i)–(iii) follow as in the proof of Proposition 2. On the balanced growth path, $X = (g+\delta)K$, so capital accumulation is equivalent to the static linear technology $K = X/(g+\delta)$ for capital services. With output as numeraire, the investment good has price one and the rental rate is $R = r + \delta$, so the ratio of capital rental income to investment expenditure is $(r+\delta)/(g+\delta) = \mu^K$. The resulting profit of the capital-service industry, $RK - X = (r-g)K$, when rebated to the household, reproduces the balanced-growth budget constraint. For part (iv), the firm's first-order condition equates the rental rate to the private marginal product of capital; evaluating at $K = \bar{K} = K_0$ gives $\mu(r+\delta) = \text{MPK}$ with MPK as in (51), and substituting $r = \rho + g/\gamma$ yields (52). Finally, the static equilibrium reproduces the scale of the balanced growth path: given $\bar{K} = K_0$, the price of capital services is $\mu^K \cdot (g+\delta) = r + \delta$, and since the private marginal product is strictly decreasing in $K/(K_0 L)$, the first-order condition $\mu(r+\delta) = \text{MPK}(K/(K_0 L))$ has the unique solution $K = K_0$. □

Relative to Proposition 2, the only new ingredient is part (iv): a single additional equation determines the growth rate, and the capital-services productivity A^K and capital wedge μ^K are then evaluated at the resulting g. The growth rate is pinned down by the capital cost share and the capital-output ratio, both observable on the initial balanced growth path; the technology (50) matters only through how these statistics respond to shocks. Shocks that change the growth rate—for example, a change in A or L, which moves the marginal product in (51)—also change the detrending rate, so the new balanced growth path is not parallel to the initial one. This means that it is not possible to do the level comparison performed in the main text, but the equivalence result nonetheless characterizes outcomes such as capital shares for each balanced growth path.

B.2.4 Biased Productivity Growth.

This appendix gives the simplest version of the biased-growth extension. There is one country, one labor input, one capital good, and no capital taxes. A final good is produced from labor and capital,

$$Y = A_Y L^{1-\alpha} K^{\alpha}, \qquad \alpha \in (0,1), \tag{53}$$

and is transformed into consumption and investment goods according to

$$C = A^C Y^C, \qquad X = A_X Y^X, \qquad Y^C + Y^X = Y. \tag{54}$$

Capital accumulates according to $\dot{K} = X - \delta K$. Labor grows at rate n and each A^m grows at constant rate γ_m for $m \in \{Y, C, X\}$. Consumption is the numeraire, $p^C \equiv 1$, so r is the real interest rate in consumption units. We write g_Z for the growth rate of a variable Z and π_X for the growth rate of the investment price p^X.

Proposition 14 (Static equivalence with biased consumption and investment growth)**.** *Consider an interior generalized balanced-growth path with constant r. Then:*

(i) Prices satisfy $p^Y = A^C$ and $p^X = A^C / A^X$, so that $\pi_X = \gamma_C - \gamma_X$, and the rental rate of capital is

$$R = (r + \delta - \pi_X)\, p^X. \tag{55}$$

(ii) The balanced-growth rates are

$$g_Y = n + \frac{\gamma_Y + \alpha\gamma_X}{1 - \alpha}, \qquad g_K = g_X = g_Y + \gamma_X, \qquad g_C = g_Y + \gamma_C, \tag{56}$$

which imply $\pi_X + g_K = g_C$. The investment share of final output is constant,

$$s_X \equiv \frac{Y^X}{Y} = \frac{\alpha(g_K + \delta)}{r + \delta - \pi_X}, \tag{57}$$

and the path is interior iff $0 < s_X < 1$.

(iii) At each date t, the BGP allocation is an equilibrium of a static economy with the date-t technologies (53)–(54), except that capital services are produced from investment goods with the linear technology $K = X/(g_K + \delta)$ and sold at markup

$$\mu^K = \frac{r + \delta - \pi_X}{g_K + \delta}, \tag{58}$$

with markup revenue $RK - p^X X = (r - g_C)\, p^X K$ rebated to the household.

Relative to Proposition 2, biased growth only changes the capital block: the capital-services productivity $1/(g + \delta)$ becomes $1/(g_K + \delta)$, and the capital wedge $(r + \delta)/(g + \delta)$ becomes $(r + \delta - \pi_X)/(g_K + \delta)$. The baseline formulas apply once the return is measured in investment-good units, $r - \pi_X$, and the growth rate is that of the capital stock. When $\gamma_C = \gamma_X$, the investment price is constant and $g_K = g_C$, recovering the baseline.

Proof. Zero profits in the technologies for consumption and investment give $p^Y = A^C$ and $p^X = p^Y/A^X = A^C/A^X$, so $\pi_X = \gamma_C - \gamma_X$. A standard user cost argument yields $rp^X = R + \dot{p}^X - \delta p^X$, which gives (55).

For the growth rates, the Cobb–Douglas capital share condition $RK = \alpha p^Y Y$ implies $\pi_X + g_K = \gamma_C + g_Y$, since R grows at π_X when r is constant. Hence $g_K = g_Y + \gamma_X$. Differentiating the production function, $g_Y = \gamma_Y + (1-\alpha)n + \alpha g_K$; substituting $g_K = g_Y + \gamma_X$ and solving yields the expressions in (56). On a BGP, $X = (g_K + \delta)K$, so $g_X = g_K$. The final-good input to investment, $Y^X = X/A^X$, then grows at $g_K - \gamma_X = g_Y$, so s_X is constant and $g_C = \gamma_C + g_Y$. Combining $RK = \alpha p^Y Y$ with (55) gives the level of s_X in (57), and $\pi_X + g_K = (\gamma_C - \gamma_X) + (g_Y + \gamma_X) = g_C$.

For the static equivalence, the accumulation equation $X = (g_K + \delta)K$ is the resource constraint of the linear technology $K = X/(g_K + \delta)$, whose unit cost is $(g_K + \delta)p^X$. With markup (58), the static rental price is $\mu^K(g_K + \delta)p^X = (r + \delta - \pi_X)p^X = R$, so the final-good firm faces the same wage and rental rate as in the dynamic economy; the remaining blocks are unchanged. For the household, asset-market clearing gives $B = p^X K$, so $\dot{B}/B = \pi_X + g_K = g_C$ and dynamic net capital income is $rB - \dot{B} = (r - g_C)p^X K$. This equals the static markup revenue, $RK - p^X X = (r - \pi_X - g_K)p^X K = (r - g_C)p^X K$, so the household budget constraints coincide. □

B.2.5 Labor-leisure choice

In our main exercise, we abstract from labor-leisure choice. However, adding such a choice is quite straightforward: instead of labor, households have a time endowment that can be allocated to leisure or production. This is isomorphic to there being another good called leisure produced from labor. Given this modification, the equivalence between balanced growth paths and the equivalent static economy still holds, with the equivalent static economy now also featuring a labor-leisure choice. Note that under this modification, changes in consumption now include changes in leisure. In general, the existence of a balanced growth path requires leisure to also benefit from productivity growth.

B.3 Proofs for this Section

Proof of Proposition 2. The equilibrium of the static economy in the proposition is characterized by the production functions (3)–(5), the production function for capital services $K_{hk} = X_{hk}/(g + \delta_k)$, cost minimization given output levels and those production functions, prices given by (7) and $R_{hk} = \mu^K_{hk}\,\mathcal{C}^K_{hk} = \mu^K_{hk}(g + \delta_k)\,\mathcal{C}^X_{hk}$ with $r_{hk} = \rho + g/\gamma + \tau^K_{hk}$, and the resource constraints (10)–(14). Household consumption satisfies the budget constraint

$$p_h^C C_h = \sum_{f\in\mathcal{F}} w_{hf} L_{hf} + \sum_{i\in\mathcal{N}} (1 - 1/\mu_{hi}^Y)\, p_{hi}^Y Y_{hi}^Y + \sum_{k\in\mathcal{K}} (1 - 1/\mu_{hk}^K)\, R_{hk} K_{hk} + \mathcal{T}_h, \tag{59}$$

which equates consumption expenditure to factor income, markup revenue from goods and capital services, and international transfers.

Now suppose that we have a balanced growth path of the dynamic economy. The production functions, the price equations (7), and the resource constraints coincide with those of the static economy, and the accumulation equation (6) coincides with the production function for capital services. Since $R_{hk} = (r_{hk} + \delta_k)\, p_{hk}^X$ by (8) and $p_{hk}^X = \mathcal{C}_{hk}^X$, we have

$$R_{hk} = \underbrace{\frac{r_{hk} + \delta_k}{g + \delta_k}}_{\mu_{hk}^K} (g + \delta_k)\, \mathcal{C}_{hk}^X,$$

so the price equation for capital in the static economy is also satisfied. Returns satisfy $r_{hk} = \rho + g/\gamma + \tau_{hk}^K$ by (16). It remains to verify the budget constraint. Substituting the transfer income (9) and the asset market clearing condition (15) into the balanced-growth budget constraint (2):

$$\begin{aligned}
p_h^C C_h &= \sum_{f\in\mathcal{F}} w_{hf} L_{hf} + \sum_{k\in\mathcal{K}} (r_{hk} - \tau_{hk}^K - g) B_{hk} + \left(\sum_{k\in\mathcal{K}} \tau_{hk}^K B_{hk} + \sum_{i\in\mathcal{N}} (1 - 1/\mu_{hi}^Y)\, p_{hi}^Y Y_{hi}^Y + \mathcal{T}_h \right) \\
&= \sum_{f\in\mathcal{F}} w_{hf} L_{hf} + \sum_{i\in\mathcal{N}} (1 - 1/\mu_{hi}^Y)\, p_{hi}^Y Y_{hi}^Y + \sum_{k\in\mathcal{K}} (r_{hk} - g)\, p_{hk}^X K_{hk} + \mathcal{T}_h \\
&= \sum_{f\in\mathcal{F}} w_{hf} L_{hf} + \sum_{i\in\mathcal{N}} (1 - 1/\mu_{hi}^Y)\, p_{hi}^Y Y_{hi}^Y + \sum_{k\in\mathcal{K}} \frac{r_{hk} - g}{r_{hk} + \delta_k} (r_{hk} + \delta_k)\, p_{hk}^X K_{hk} + \mathcal{T}_h \\
&= \sum_{f\in\mathcal{F}} w_{hf} L_{hf} + \sum_{i\in\mathcal{N}} (1 - 1/\mu_{hi}^Y)\, p_{hi}^Y Y_{hi}^Y + \sum_{k\in\mathcal{K}} \underbrace{\left[1 - \frac{g + \delta_k}{r_{hk} + \delta_k} \right]}_{1 - 1/\mu_{hk}^K} R_{hk} K_{hk} + \mathcal{T}_h,
\end{aligned}$$

which is the static budget constraint (59). □

Proof of Proposition 3. Let

$$S \equiv \sum_{h\in\mathcal{H}} p_h^C C_h$$

denote world nominal consumption. We use the ordering in Equation (18): consumption goods, intermediate goods, investment goods, capital goods, and labor inputs. Let $\mathcal{P}$ denote the first four blocks, i.e. all non-labor rows and columns, and let $\mathcal{W}$ denote the labor block. The vector $\mathbf{p}$ stacks the prices of the non-labor rows,

$$\mathbf{p} = \big((p_h^C)_{h\in\mathcal{H}}, (p_{hi}^Y)_{h,i}, (p_{hk}^X)_{h,k}, (R_{hk})_{h,k}\big),$$

while $\mathbf{w}$ stacks labor prices. For rows that have no direct technology shifter or no wedge, set the corresponding entries of $d\log\mathbf{A}$ or $d\log\boldsymbol{\mu}$ equal to zero. Thus the standard-goods rows contain $(d\log A_{hi}^Y, d\log\mu_{hi}^Y)$, the investment-goods rows contain $d\log A_{hk}^X$ and no wedge, and the capital-goods rows contain $d\log\mu_{hk}^K$ and no direct technology shifter. Constants such as $g+\delta_k$ drop out of log changes.

Prices. By Proposition 2, balanced-growth prices are prices in the equivalent static economy. For every non-labor producer $a \in \mathcal{P}$, the pricing equation in the equivalent static economy satisfies:

$$p_a = \mu_a A_a^{-1} \mathcal{C}_{F_a}(\mathbf{p}, \mathbf{w}),$$

where $\mathcal{C}_{F_a}$ is the unit cost function associated with the production function F_a. This notation includes the capital-service rows: for capital good (h,k), the unit cost of producing one unit of capital services is $\mathcal{C}_{hk}^K = (g+\delta_k)\mathcal{C}_{hk}^X$, so the capital row of $\tilde{\Omega}$ places unit weight on the corresponding investment-good price p_{hk}^X. Log-linearizing and applying Shephard's lemma gives, for every $a \in \mathcal{P}$,

$$d\log p_a = -d\log A_a + d\log\mu_a + \sum_{b\in\mathcal{P}} \tilde{\Omega}_{a,b}\, d\log p_b + \sum_{\ell\in\mathcal{W}} \tilde{\Omega}_{a,\ell}\, d\log w_\ell.$$

Stacking these equations over all non-labor rows gives

$$d\log\mathbf{p} = -d\log\mathbf{A} + d\log\boldsymbol{\mu} + \tilde{\Omega}_{PP} d\log\mathbf{p} + \tilde{\Omega}_{PW} d\log\mathbf{w}, \tag{60}$$

which is (20). Written out by blocks, (60) contains the consumption-good price equations, the intermediate good price equations, the investment-good price equations, and the capital-service price equations

$$d\log R_{hk} = d\log\mu_{hk}^K + d\log p_{hk}^X.$$

For a capital-tax perturbation, the after-tax return is fixed by preferences on the balanced growth path, so the pre-tax return changes one-for-one with the additive capital tax: $dr_{hk} = d\tau_{hk}^K$. Since

$$\mu_{hk}^K = \frac{r_{hk}+\delta_k}{g+\delta_k},$$

we obtain

$$d \log \mu^K_{hk} = \frac{d\tau^K_{hk}}{r_{hk} + \delta_k}, \tag{61}$$

which is the expression in the proposition.

Cost shares. By Shephard's lemma, we have that $y_{ab} = \partial C_a / \partial p_b$, where C_a is the cost function of a. Totally differentiating this gives:

$$d \log y_{ab} = \sum_c \theta^a_{bc} d \log p_c + \sum_d \theta^a_{bd} d \log w_d + d \log y_a - d \log A_a,$$

where we use θ^a_{bc} as a shorthand for cross-price elasticity of demand. Using the identity $\sigma^a_{bc} = \theta^a_{bc} / \tilde{\Omega}_{ac}$, we can write:

$$d \log y_{ab} = \theta^a_{bb} d \log p_b + \sum_{c \neq b} \sigma^a_{bc} \tilde{\Omega}_{ac} d \log p_c + \sum_{d \neq b} \sigma^a_{bd} \tilde{\Omega}_{ad} d \log w_d + d \log y_a - d \log A_a,$$

Using the definition $\tilde{\Omega}_{ab} = p_b y_{ab} / [y_a A_a^{-1} \mathcal{C}_{F_a}]$, we get

$$\begin{aligned} d \log \tilde{\Omega}_{ab} = {} & d \log p_b + \theta^a_{bb} d \log p_b - \tilde{\Omega}_{ab} d \log p_b + \sum_{c \neq b} \sigma^a_{bc} \tilde{\Omega}_{ac} d \log p_c + \sum_{d \neq b} \sigma^a_{bd} \tilde{\Omega}_{ad} d \log w_d \\ & - \sum_{c \neq b} \tilde{\Omega}_{ac} d \log p_c - \sum_{d \neq b} \tilde{\Omega}_{ad} d \log w_d \end{aligned}$$

where we use Shephard's lemma to express $d \log \mathcal{C}_{F_a}$ in terms of changes in input prices. By homogeneity of degree zero of the demand function in input prices, we know that

$$\theta^a_{bb} = - \sum_{c \neq b} \theta^a_{bc},$$

where c ranges over all other inputs (both intermediate inputs and labor). Hence, for all pairs with $\tilde{\Omega}_{ab} \in (0, 1)$, we have:

$$\begin{aligned} d \log \tilde{\Omega}_{ab} = {} & d \log p_b - \left[\sum_{c \neq b} \theta^a_{bc} \right] d \log p_b - \tilde{\Omega}_{ab} d \log p_b + \sum_{c \neq b} \sigma^a_{bc} \tilde{\Omega}_{ac} d \log p_c - \sum_{c \neq b} \tilde{\Omega}_{ac} d \log p_c \\ & + \sum_{d \neq b} \tilde{\Omega}_{ad} \sigma^a_{bd} d \log w_d - \sum_{d \neq b} \tilde{\Omega}_{ad} d \log w_d \\ = {} & d \log p_b - \frac{\left[\sum_{c \neq b} \theta^a_{bc} \right]}{\tilde{\Omega}_{ab}} \tilde{\Omega}_{ab} d \log p_b - \tilde{\Omega}_{ab} d \log p_b \\ & + \sum_{c \neq b} \left[\sigma^a_{bc} - 1 \right] \tilde{\Omega}_{ac} d \log p_c + \sum_{d \neq b} \left[\sigma^a_{bd} - 1 \right] \tilde{\Omega}_{ad} d \log w_d \end{aligned}$$

$$= \left[1-\tilde{\Omega}_{ab}\right]\left[\left[1-\bar{\sigma}^a_b\right] d\log p_b - \sum_{c\neq b}\left[1-\sigma^a_{bc}\right]\frac{\tilde{\Omega}_{ac}}{\left[1-\tilde{\Omega}_{ab}\right]} d\log p_c \right.$$
$$\left. - \sum_{d\neq b}\left[1-\sigma^a_{bd}\right]\frac{\tilde{\Omega}_{ad}}{\left[1-\tilde{\Omega}_{ab}\right]} d\log w_d \right].$$

This is (21). When b is a labor input, interpret $d\log p_b = d\log w_b$.

Industry sizes. Let $\lambda_a = p_a Y_a / S$ denote sales of row a relative to world consumption, with labor sales given by $\lambda_{hf} = w_{hf}L_{hf}/S$. Let $\mathbf{\Phi}$ be the vector of final shares padded with zeros outside the consumption-good block. Sales of any good b equal final sales to households plus sales to downstream producers:

$$\lambda_b = \Phi_b + \sum_a \frac{\lambda_a}{\mu_a}\tilde{\Omega}_{a,b}.$$

Totally differentiating this expression yields (22).

Consumption shares. Dividing country h's balanced-growth household budget constraint by S, normalizing $S=1$, yields:

$$\Phi_h = \sum_{f\in\mathcal{F}} \lambda_{hf} + \sum_{k\in\mathcal{K}} \lambda_{hk}\left(1-\frac{1}{\mu^K_{hk}}\right) + \sum_{i\in\mathcal{N}} \lambda_{hi}\left(1-\frac{1}{\mu^Y_{hi}}\right) + \mathcal{T}_h. \tag{62}$$

Differentiating gives

$$d\Phi_h = \sum_{f\in\mathcal{F}} d\lambda_{hf} + \sum_{k\in\mathcal{K}} d\left[\lambda_{hk}\left(1-\frac{1}{\mu^K_{hk}}\right)\right] + \sum_{i\in\mathcal{N}} d\left[\lambda_{hi}\left(1-\frac{1}{\mu^Y_{hi}}\right)\right] + d\mathcal{T}_h. \tag{63}$$

Since world nominal consumption S is used as the numeraire, $d\log S = 0$. Since labor supplies are fixed in balanced-growth units, $d\lambda_{hf} = \lambda_{hf} d\log w_{hf}$, and (63) becomes

$$d\Phi_h = \sum_{f\in\mathcal{F}} \lambda_{hf} d\log w_{hf} + \sum_{k\in\mathcal{K}} d\left[\lambda_{hk}\left(1-\frac{1}{\mu^K_{hk}}\right)\right] + \sum_{i\in\mathcal{N}} d\left[\lambda_{hi}\left(1-\frac{1}{\mu^Y_{hi}}\right)\right] + d\mathcal{T}_h. \tag{64}$$

Since $d\mathcal{T}_h = 0$ under the assumed shock, this is (23). Equivalently, if the numeraire is left explicit, the first term in (23) can be replaced by $\sum_f d\lambda_{hf}$, or by $\sum_f \lambda_{hf}(d\log w_{hf} - d\log S)$.

Labor market clearing and quantity recovery. For each labor type,

$$\lambda_{hf} = \frac{w_{hf}L_{hf}}{S}.$$

Since L_{hf} is fixed in balanced-growth units,

$$0 = d\log L_{hf} = d\log\lambda_{hf} - \left(d\log w_{hf} - d\log S\right), \tag{65}$$

which is (24). More generally, for any produced good or factor a, the definition $\lambda_a = p_a Y_a / S$ implies

$$d\log Y_a = d\log\lambda_a - \left(d\log p_a - d\log S\right), \tag{66}$$

with $p_a = w_{hf}$ for labor inputs. Thus, once the system solves for prices, wages, shares, and wedges, balanced-growth quantities are recovered from (66). Bilateral quantities can be obtained by noting that for any non-consumption good, the physical amount sent from b to a satisfies $x_{a,b} = x_b \frac{\lambda_a/\mu_a\tilde{\Omega}_{a,b}}{\sum_{a'}\lambda_{a'}/\mu_{a'}\tilde{\Omega}_{a',b}}$, which means $d\log x_{ab}$ is pinned down given $d\log x_b$, $d\log\lambda_a$, $d\log\mu_a$, and $d\log\tilde{\Omega}_{a,b}$.

Equations (60)–(65), together with (61) for capital-tax shocks, are exactly the system in Proposition 3. The coefficients in the price equation are entries of $\tilde{\Omega}$; the derivatives of cost shares are pinned down by $\boldsymbol{\sigma}$; revenue shares are pinned down by $\tilde{\Omega}$ and $\boldsymbol{\mu}$; and $\boldsymbol{\lambda}$ is pinned down by $\boldsymbol{\Phi}$, $\tilde{\Omega}$, and $\boldsymbol{\mu}$ through (19) and (23). Capital-tax shocks additionally require r_{hk} and δ_k through (61). $\square$

C Appendix to Section 5: Global Consumption Responses

C.1 Technological versus reallocation effects

Define an *allocation matrix* $\mathcal{X}_{a,b} = \frac{Y_{a,b}}{Y_b}$ as the share of b's physical output that is directed to industry a. Analogously, define $\mathcal{X}_{a,hf} = L_{a,hf}/L_{hf}$ as the share of labor L_{hf} that is directed to industry a. The columns of $\mathcal{X}$ sum to one for all industries but for consumption goods, where they sum to zero. This captures that all industries apart from consumption goods are sent to another industry, while no industry uses consumption goods as inputs.

The allocation matrix implicitly defines an allocation via the solution to equation

$$Y_a = A_a F_a[\mathcal{X}_{a,b_1}Y_{b_1}, \ldots, \mathcal{X}_{a,b_P}Y_{b_P}, \mathcal{X}_{a,f_1}L_{f_1}, \ldots, \mathcal{X}_{a,HF}L_{f_{HF}}], \tag{67}$$

where we use Y_a and $Y_{a,b}$ to denote outputs of all non-factor industries.[35] We define the

[35] Given an observed feasible allocation, the allocation matrix $\mathcal{X}$ derived from that allocation will automatically solve this equation. Global uniqueness is harder to establish in general. However, the solution will typically be locally unique, and this is possible to verify using standard implicit function criteria. Local uniqueness is sufficient for comparative statics under an additional assumption that the nearby equilibrium is selected after small perturbations to parameters.

technological effect as the change in aggregate consumption resulting from modifying A_a in this equation leaving $\mathcal{X}$ invariant. To do this, we first note that the changes in productivities are

$$\begin{aligned} d\log Y_a &= d\log A_a + \sum_b \frac{\partial \log F_a}{\partial \log Y_b} d\log[\mathcal{X}_{a,b} Y_b] \\ &= d\log A_a + \sum_b \tilde{\Omega}^{PP}_{a,b} d\log Y_b. \end{aligned}$$

Hence, we have

$$d\log \boldsymbol{Y} = (I - \tilde{\Omega}^{PP})^{-1} d\log \boldsymbol{A}.$$

Furthermore, since the consumption good exclusively goes to consumption, we obtain

$$\sum_h \Phi_h d\log Y_h = \sum_i \left[\sum_h \Phi_h (I - \tilde{\Omega}^{PP})^{-1}_{hi}\right] d\log \boldsymbol{A}_i = \sum_i \tilde{\lambda}_i d\log A_i.$$

So $\tilde{\lambda}_i$ is the technological effect of changing A_i. Since the full effect on consumption is obtained by totally differentiating (67), we obtain that the remaining term in the proposition is the partial derivative of global consumption with respect to $\mathcal{X}$.

C.2 Details on Example 2

In our formalism, the full technology of the neoclassical model is

$$C = Y_C, \quad Y = A_Y \left[(1-\alpha)^{\frac{1}{\sigma_{KL}}} L^{\frac{\sigma_{KL}-1}{\sigma_{KL}}} + \alpha^{\frac{1}{\sigma_{KL}}} K^{\frac{\sigma_{KL}-1}{\sigma_{KL}}}\right]^{\frac{\sigma_{KL}}{\sigma_{KL}-1}}, \quad X = Y_X = (g+\delta)K, \quad Y = Y_C + Y_X$$

The cost weight of sector Y satisfies the following system of equations

$$\tilde{\lambda}_C = 1, \qquad \tilde{\lambda}_Y = \tilde{\lambda}_C + \tilde{\lambda}_X, \qquad \tilde{\lambda}_X = \tilde{\lambda}_K, \qquad \tilde{\lambda}_K = \tilde{\Omega}_{Y,K} \tilde{\lambda}_Y,$$

which gives us $\tilde{\lambda}_Y = 1/(1 - \tilde{\Omega}_{Y,K})$. Writing $\alpha = \tilde{\Omega}_{Y,K}$ for the initial value of the capital cost share, we obtain the technological effect of the productivity change $d\log A_Y$.

To obtain the change in the labor share, we note that it satisfies the following set of equations

$$\lambda_L = (1-\alpha)\lambda_Y, \quad \lambda_K = \alpha\lambda_Y, \quad \lambda_Y = \lambda_C + \lambda_X, \lambda_X = \left(\frac{r+\delta}{g+\delta}\right)^{-1} \lambda_K, \quad \lambda_C = 1$$

This implies

$$d\log\lambda_L = d\log(1-\tilde{\Omega}_{Y,K}) + d\log\lambda_Y = -\frac{\alpha}{1-\alpha}d\log\tilde{\Omega}_{Y,K} + d\log\lambda_Y$$

It is helpful to express things in terms of the share of output that is dedicated to investment, $x \equiv \lambda_X/\lambda_Y$. In particular, we have

$$d\log\lambda_Y = xd\log\lambda_X \iff d\log\lambda_Y = \frac{x}{1-x}d\log\frac{\lambda_X}{\lambda_Y} = \lambda_X d\log\frac{\lambda_X}{\lambda_Y}.$$

Furthermore, we have

$$d\log\frac{\lambda_X}{\lambda_Y} = d\log\frac{\tilde{\Omega}_{Y,K}\lambda_Y}{\lambda_Y} = d\log\tilde{\Omega}_{Y,K}.$$

This means that

$$d\log\lambda_L = \left(\lambda_X - \frac{\alpha}{1-\alpha}\right)d\log\tilde{\Omega}_{Y,K}.$$

Intuitively, a higher α lowers the labor share insofar as the investment share $\lambda_X < \frac{\alpha}{1-\alpha}$, which happens if $\mu > 1$, since

$$\lambda_X = \mu^{-1}\alpha\lambda_Y = \mu^{-1}\alpha(1+\lambda_X) \iff \lambda_X = \frac{\mu^{-1}\alpha}{1-\mu^{-1}\alpha}.$$

To obtain the final result, we note that

$$\lambda_X - \frac{\alpha}{1-\alpha} = -\lambda_X\frac{1}{1-\alpha}\frac{r-g}{g+\delta}.$$

Furthermore, we have $d\log\tilde{\Omega}_{Y,K} = (\sigma_{KL}-1)d\log A_Y$ since

$$d\log\tilde{\Omega}_{Y,K} = -(\sigma_{KL}-1)(1-\alpha)d\log R/w = (\sigma_{KL}-1)d\log A_Y.$$

Using that

$$d\log C = \tilde{\lambda}_Y d\log A_Y - d\log\lambda_L,$$

we obtain the result.

C.3 Details on Example 3: Effects of a Tariff

By symmetry, we can derive everything within one country and assume that changes in foreign prices equal changes in domestic prices. Given that, the effect of changing a tariff is characterized by the following equations

$$
\begin{aligned}
d\log C &= \lambda_X \frac{r-g}{g+\delta} d\log K,\\
d\log K = d\log \frac{K}{L} &= -\sigma_{KL} d\log \frac{R}{w},\\
d\log R &= d\log p_X = d\log p_Y,\\
d\log p_Y &= (1-\beta) d\log p_Q + \beta d\log p_{imp},\\
d\log p_{imp} &= d\log p_Q + d\log(1+t),\\
d\log p_Q &= \alpha d\log R + (1-\alpha) d\log w.
\end{aligned}
$$

The first equation uses that the tariff change is a distortion shock and applies Proposition 5, where $\lambda_X = \frac{p_X X}{p_C C}$ is investment expenditure relative to consumption in each country.[36] The second equation uses firm optimality and that labor supply is fixed, while the third equation uses that investment is produced by the final output good and that rates of return do not change. We then express changes in the final output good in terms of changes in the local variety and import prices, and use symmetry to conclude import prices change in line with domestic prices plus the tariff. The last equation expresses changes in the domestic variety in terms of capital rental rates and wages.

Putting the equations together, we get that

$$
\begin{aligned}
d\log K &= -\sigma_{KL} d\log \frac{R}{w}\\
&= -\sigma_{KL} d\log \frac{p_Y}{w}\\
&= -\sigma_{KL} [d\log \frac{p_Q}{w} + \beta d\log(1+t)]\\
&= -\sigma_{KL} [\alpha d\log \frac{R}{w} + \beta d\log(1+t)]\\
&= -\sigma_{KL} [-\alpha \frac{d\log K}{\sigma_{KL}} + \beta d\log(1+t)]
\end{aligned}
$$

which yields

$$
d\log K = -\frac{\beta}{1-\alpha} \times \sigma_{KL} d\log(1+t),
$$

which yields the proposition via the expression for $d\log C$.

[36] More precisely, the proposition gives us that global consumption is $\sum_{h=1}^{2} \lambda_{X_h} \frac{r-g}{g+\delta} d\log K_h$, by symmetry, we can focus on outcomes in one country.

C.4 General formula for changes in global GDP

The following result gives the comparative statics of long-run GDP in the general case without Cobb-Douglas production functions. Using this result, Appendix C.5 supplies the proof of Proposition 7.

Proposition 15 (Productivity Shocks and GDP)**.** *Under the assumptions of Proposition 4, the long-run elasticity of real GDP with respect to productivity A^Y_{hi} is*

$$\begin{aligned}\frac{d\log Y}{d\log A^Y_{hi}} &= \tilde{\lambda}_{Y_{hi}} - \sum_{f\in\mathcal{F}} \tilde{\lambda}_f \frac{d\log\lambda_f}{d\log A^Y_{hi}} \\ &\quad + s_X\left[\left(\tilde{\lambda}^X_{Y_{hi}} - \tilde{\lambda}_{Y_{hi}}\right) - \sum_{f\in\mathcal{F}}\left(\tilde{\lambda}^X_f - \tilde{\lambda}_f\right)\frac{d\log\lambda_f}{d\log A^Y_{hi}} + \frac{d\log\lambda_X}{d\log A^Y_{hi}}\right].\end{aligned}$$

The first line is the effect on consumption. In the second line, the first two terms capture the relative exposure of investment and consumption costs to the productivity shock, while the final term captures reallocation of expenditure toward or away from investment.

Proof. Start from the GDP decomposition,

$$d\log Y = d\log C + \frac{s_X}{1-s_X} d\log\frac{X}{Y} = d\log C + s_X d\log\frac{X}{C}.$$

Writing

$$\lambda_X = \frac{\sum_{h\in\mathcal{H},k\in\mathcal{K}} P^X_{hk} X_{hk}}{\sum_{h\in\mathcal{H}} P^C_h C_h},$$

for investment expenditure relative to consumption, we have

$$d\log\frac{X}{C} = d\log\lambda_X - d\log\frac{P_X}{P_C},$$

where $d\log P_X = \sum_{h\in\mathcal{H},k\in\mathcal{K}} \Phi^X_{hk} d\log p^X_{hk}$. The price indices for investment and consumption satisfy

$$d\log P_X = -\sum_j \tilde{\lambda}^X_j d\log A_j + \sum_f \tilde{\lambda}^X_f d\log w_f,$$

and

$$d\log P_C = -\sum_j \tilde{\lambda}_j d\log A_j + \sum_f \tilde{\lambda}_f d\log w_f.$$

Therefore, for a shock to A^Y_{hi},

$$\frac{d\log(X/C)}{d\log A^Y_{hi}} = \tilde{\lambda}^X_{Y_{hi}} - \tilde{\lambda}_{Y_{hi}} - \sum_f \left(\tilde{\lambda}^X_f - \tilde{\lambda}_f\right)\frac{d\log\lambda_f}{d\log A^Y_{hi}} + \frac{d\log\lambda_X}{d\log A^Y_{hi}},$$

where we used $d\log w_f = d\log\lambda_f$ under the normalization of aggregate nominal consumption. Combining this expression with

$$\frac{d\log C}{d\log A^Y_{hi}} = \tilde{\lambda}_{Y_{hi}} - \sum_f \tilde{\lambda}_f \frac{d\log\lambda_f}{d\log A^Y_{hi}}$$

from Proposition 4 yields the result. □

C.5 Proofs for this Section

Proof of Proposition 4. First, we note that if we divide $\tilde{\Omega}$ into non-factors and factors:

$$\tilde{\Omega} = \begin{pmatrix} \tilde{\Omega}^{PP} & \tilde{\Omega}^{PF} \\ \mathbf{0}_{|\mathcal{F}|,|\mathcal{P}|} & \mathbf{0}_{|\mathcal{F}|,|\mathcal{F}|} \end{pmatrix},$$

then we have

$$\tilde{\Psi} = \begin{pmatrix} (I-\tilde{\Omega}^{PP})^{-1} & (I-\tilde{\Omega}^{PP})^{-1}\tilde{\Omega}^{PF} \\ \mathbf{0}_{|\mathcal{F}|,|\mathcal{P}|} & I_{|\mathcal{F}|,|\mathcal{F}|} \end{pmatrix}.$$

Hence, we have $\tilde{\lambda}_i = \sum_h \Phi_h (I-\tilde{\Omega}^{PP})^{-1}_{hi}$ for all non-factors, and $\tilde{\lambda}_{hf} = \sum_i \tilde{\lambda}_i \tilde{\Omega}^{PF}_{i,hf}$ for all factors. Furthermore, using the representation in Proposition 3 and the fact that we only shock productivities, we obtain

$$d\log p_h = -\sum_i (I-\tilde{\Omega}^{PP})^{-1}_{hi} d\log A_i + \sum_{i,(h',f)} (I-\tilde{\Omega}^{PP})^{-1}_{h,i}\tilde{\Omega}^{PF}_{i,h'f} d\log w_{h'f}.$$

Last, we note that since nominal world consumption is normalized to 1 in the representation, we have that the change in real global consumption is the decrease in the price index of global consumption. Hence

$$\begin{aligned} d\log C &= -\sum_h \Phi_h d\log p_h \\ &= \sum_h \sum_i \Phi_h (I-\tilde{\Omega}^{PP})^{-1}_{hi} d\log A_i - \sum_h \sum_{i,(h',f)} \Phi_h (I-\tilde{\Omega}^{PP})^{-1}_{h,i}\tilde{\Omega}^{PF}_{i,h'f} d\log w_{h'f}. \\ &= \sum_i \tilde{\lambda}_i d\log A_i - \sum_{h'f} \tilde{\lambda}_{h'f} d\log w_{h'f}. \end{aligned}$$

Using the market clearing condition for labor $d\log\lambda_{hf} = d\log w_{hf}$ from Proposition 3, we obtain the proposition. $\square$

Proof of Corollary 1. Starting from the equation system in Proposition 3, we note that with Cobb-Douglas, cost shares $\tilde{\Omega}_{ij}$ are invariant to prices. This means that $d\log\tilde{\Omega}_{ij} = 0$. Since wedges also do not change, it can be seen that $d\boldsymbol{\lambda} = d\boldsymbol{\Phi} = 0$ and $d\log\boldsymbol{w} = 0$ solve the equations (22)-(24) when $d\tilde{\Omega} = 0$. With $d\boldsymbol{\lambda} = 0$, we obtain

$$\frac{\partial \log C}{\partial \log A_i} = \tilde{\lambda}_i$$

from Proposition 4, which establishes the corollary. $\square$

Proof of Proposition 5. If we let a and b index all non-factor industries, we have that

$$\sum_a p_a dC_a = \sum_a p_a [dY_a - \sum_b dX_{b,a}],$$

where $X_{b,a}$ is the real input of a into industry b. Note that in our setup, only the consumption goods are ultimately consumed, so the term on the right-hand side will be zero for all non-consumption goods. Further, the final consumption goods are not used as intermediate goods.

We then use that absent changes in productivities, we have

$$dY_a = \sum_b \frac{\partial A_a F_a}{\partial X_{a,b}} dX_{a,b} + \sum_{hf} \frac{\partial A_a F_a}{\partial L_{a,hf}} dL_{a,hf},$$

where b sums over non-labor inputs, and hf sums over labor inputs. Furthermore, first-order conditions with respect to inputs yield

$$\frac{1}{\mu_a} p_a \frac{\partial (A_a F_a)}{\partial X_{a,b}} = p_b \qquad \frac{1}{\mu_a} p_a \frac{\partial (A_a F_a)}{\partial L_{a,hf}} = w_{hf}.$$

Given this, we have

$$\begin{aligned}
\sum_a p_a dC_a &= \sum_a p_a [dY_a - \sum_b dX_{b,a}] \\
&= \sum_a (1 - 1/\mu_a) p_a dY_a + \sum_a \left[\frac{p_a}{\mu_a} dY_a - p_a \sum_b dX_{b,a} \right] \\
&= \sum_a (1 - 1/\mu_a) p_a dY_a + \sum_a \left[\sum_b p_b dX_{a,b} + \sum_{hf} w_{hf} dL_{a,hf} - p_a \sum_b dX_{b,a} \right]
\end{aligned}$$

$$= \sum_a (1 - 1/\mu_a) p_a dY_a,$$

where we use that the intermediate input terms cancel, and $\sum_a dL_{a,hf} = 0$ for all hf by labor market clearing. Using that the only wedges with $\mu \neq 1$ are μ^Y_{hi} and μ^K_{hk}, we get

$$d\log C = \sum_h \left[\sum_i (1 - 1/\mu^Y_{hi}) \underbrace{\frac{p^Y_{hi} Y_{hi}}{\sum_{h'} p^C_{h'} C_{h'}}}_{\lambda_{Y_{hi}}} d\log Y_{hi} + \sum_k (1 - 1/\mu^K_{hk}) \underbrace{\frac{R_{hk} K_{hk}}{\sum_{h'} p^C_{h'} C_{h'}}}_{\lambda_{K_{hk}}} d\log K_{hk} \right],$$

which is the desired result. □

Proof of Proposition 7. To prove Proposition 7, use Proposition 15, and note that if every production function and consumption aggregator is Cobb-Douglas, expenditure shares are constant, so

$$d\log \lambda_f = 0 \qquad \text{and} \qquad d\log \lambda_X = 0.$$

The proposition therefore immediately gives

$$\frac{d\log Y}{d\log A_{hi}} = \tilde{\lambda}_{Y_{hi}} + s_X \left(\tilde{\lambda}^X_{Y_{hi}} - \tilde{\lambda}_{Y_{hi}} \right).$$

Moreover, since

$$d\log \frac{X}{Y} = (1 - s_X) d\log \frac{X}{C},$$

we obtain

$$\frac{d\log(X/Y)}{d\log A_{hi}} = (1 - s_X) \left(\tilde{\lambda}^X_{Y_{hi}} - \tilde{\lambda}_{Y_{hi}} \right),$$

which is the desired result. □

D Appendix to Section 6: Country-Level Consumption Responses

D.1 Full Country-Level Responses to Productivities and Wedges

Let

$$\tilde{\boldsymbol{\lambda}}^{C_h} = (I - \tilde{\Omega})^{-1}_{C_h,\cdot}.$$

denote the row of the Leontief inverse associated with the consumption good of country h. The element $\tilde{\lambda}_a^{C_h}$ measures the exposure of the cost of h's consumption to industry or factor a, directly and indirectly through the production and investment networks. The global cost weight from Section 5 is the consumption-weighted aggregate $\tilde{\lambda}_a = \sum_{h \in \mathcal{H}} \Phi_h \tilde{\lambda}_a^{C_h}$.

Consider a change in productivity A_{hi}, holding wedges fixed and maintaining the normalization $d\log(\sum_{h'} p_{h'}^C C_{h'}) = 0$ from Proposition 3. The response of country h's consumption is

$$\frac{d\log C_h}{d\log A_{hi}^Y} = \underbrace{\tilde{\lambda}_{Y_{hi}}^{C_h}}_{\Delta \text{ technology}} + \underbrace{\sum_{f\in\mathcal{F}} \left(\frac{\lambda_{L_{hf}}}{\Phi_h} - \tilde{\lambda}_{L_{hf}}^{C_h}\right) \frac{d\log \lambda_{L_{hf}}}{d\log A_{hi}^Y} - \sum_{h'\neq h}\sum_{f\in\mathcal{F}} \tilde{\lambda}_{L_{h'f}}^{C_h} \frac{d\log\lambda_{L_{h'f}}}{d\log A_{hi}} + \frac{1}{\Phi_h}\frac{d\Pi_h}{d\log A_{hi}^Y}}_{\Delta \text{ reallocation}},$$

where

$$\Pi_h = \sum_{j\in\mathcal{N}} \left(1 - \frac{1}{\mu_{hj}^Y}\right)\lambda_{Y_{hj}} + \sum_{k\in\mathcal{K}}\left(1 - \frac{1}{\mu_{hk}^K}\right)\lambda_{K_{hk}}$$

is country h's wedge income as a share of world consumption.

To derive the result, note that under the chosen normalization $d\log C_h = d\log\Phi_h - d\log p_h^C$. Shephard's lemma implies $d\log p_h^C = -\tilde{\lambda}_{Y_{hi}}^{C_h} d\log A_{hi}^Y + \sum_{h',f} \tilde{\lambda}_{L_{h'f}}^{C_h} d\log w_{h'f}$, the income change $d\Phi_h$ is given by (23), and changes in factor income equal changes in factor prices by (24).

The reallocation terms compare income and cost exposures. The first sum weights domestic labor income by the difference between its share of h's income, $\lambda_{L_{hf}}/\Phi_h$, and its share of h's consumption costs, $\tilde{\lambda}_{L_{hf}}^{C_h}$; the second captures h's cost exposure to foreign factors; the third captures changes in wedge income. Unlike the global response in Proposition 4, these terms are first-order even at the Golden Rule, since income shares and cost exposures differ country by country through trade. Weighting the country responses above by Φ_h and summing over countries, the income terms cancel against the cost exposures ($\sum_h d\Phi_h = 0$) and the expression collapses to Proposition 4.

D.2 Definition of a Small Open Economy

Given a vector of import prices $\{p_{h'i}^Y\}_{h'\neq h, i\in\mathcal{N}}$ and export demand schedules (28), a balanced growth path in a small open economy is a set of country h quantities and prices such that the balanced growth path equations (2)-(16) associated with country h are satisfied and the export demand equations (28) are satisfied.

D.3 Proofs for this section

Proof of Proposition 8. The assumption of balanced trade implies that initial transfers are zero $\mathcal{T}_h = 0$.[37] A balanced growth path of the small open economy is a solution to the BGP equations (2)–(16) associated with country h, together with the export demand schedules (28). We first write down the equations of the equivalent static economy, and then show that all those equations are satisfied by a dynamic small open economy equilibrium.

Equations of the equivalent static economy. The equations associated with conditions (i)–(iii) of Proposition 8 are the same as in the proof of Proposition 2: the production functions (3)–(5), the pricing equations (7), capital services produced from investment goods with productivity $A^K_{hk} = 1/(g+\delta_k)$ and sold at the markup μ^K_{hk}, and the resource constraints (10)–(14), restricted to country h. Conditions (iv)–(v) add a trade block. The dollar good is produced from export goods with the decreasing-returns technology

$$Y_{h\$} = F_{h\$}\left[\{Y^{exp}_{hi}\}_{i\in\mathcal{N}}\right] \equiv \sum_{i\in\mathcal{N}} D^{\frac{1}{\theta}}_{hi}\left(Y^{exp}_{hi}\right)^{1-\frac{1}{\theta}}. \tag{68}$$

Let $\mathcal{C}_\$(p^Y_{h1},\ldots,p^Y_{hN};Y_{h\$})$ denote the associated cost function. Since $F_{h\$}$ has decreasing returns, marginal cost depends on scale, and cost minimization determines input quantities and the pricing condition separately. Export goods are the cost-minimizing input quantities,

$$Y^{exp}_{hi} = \frac{\partial \mathcal{C}_\$}{\partial p^Y_{hi}}(p^Y_{h1},\ldots,p^Y_{hN};Y_{h\$}), \qquad i\in\mathcal{N}, \tag{69}$$

and the price of the dollar, which we normalize to one, equals marginal cost times the subsidy wedge $\mu^Y_{h\$} = 1-1/\theta$:

$$1 = \left(1-\frac{1}{\theta}\right)\frac{\partial \mathcal{C}_\$}{\partial Y_\$}(p^Y_{h1},\ldots,p^Y_{hN};Y_{h\$}). \tag{70}$$

Each import good $(h'j)$ with $h' \neq h$ is produced from dollars with a linear technology with productivity $1/p^Y_{h'j}$, so its price equals $p^Y_{h'j}$. The transfer to the household adds the profits of the dollar sector, $\Pi_{h\$} = (1-1/\theta)^{-1}Y_{h\$} - \mathcal{C}_\$$, and subtracts the lump-sum cost

[37] The case $\mathcal{T}_h \neq 0$ requires endowing the household in the equivalent static economy with $\mathcal{T}_h$ units of the dollar good; the argument is otherwise unchanged.

of financing the subsidy, $S_{h\$} = \left[(1-1/\theta)^{-1} - 1\right] Y_{h\$}$:

$$T_h = \sum_{k\in\mathcal{K}} \tau^K_{hk} B_{hk} + \sum_{i\in\mathcal{N}} (1 - 1/\mu^Y_{hi})\, p^Y_{hi} Y^Y_{hi} + \Pi_{h\$} - S_{h\$}. \tag{71}$$

Finally, market clearing for dollars requires that dollar production equals its use in producing imports,

$$Y_{h\$} = \sum_{h'\neq h} \sum_{j\in\mathcal{N}} p^Y_{h'j} M_{h'j}, \qquad M_{h'j} \equiv Y^C_{h,h'j} + \sum_{i\in\mathcal{N}} Y^Y_{hi,h'j} + \sum_{k\in\mathcal{K}} Y^X_{hk,h'j}, \tag{72}$$

where $M_{h'j}$ is the domestic use of the imported good $(h'j)$.

Two properties of the trade block are used repeatedly. First, since $F_{h\$}$ is homogeneous of degree $1-1/\theta$, the cost function satisfies $Y_\$\, \partial\mathcal{C}_\$/\partial Y_\$ = (1-1/\theta)^{-1}\mathcal{C}_\$$, so the pricing condition (70) implies

$$\mathcal{C}_\$ = Y_{h\$}, \tag{73}$$

stating that total spending on export goods equals dollar output. It follows that $\Pi_{h\$} = S_{h\$} = Y_{h\$}/(\theta-1)$, so the two new terms in (71) cancel and the transfer coincides with its balanced-growth counterpart (9). Second, the first-order conditions of cost minimization state $p^Y_{hi} = \xi\,(1-1/\theta)\, D^{1/\theta}_{hi} (Y^{exp}_{hi})^{-1/\theta}$ with multiplier $\xi = \partial\mathcal{C}_\$/\partial Y_\$$; substituting $\xi = (1-1/\theta)^{-1}$ from (70) yields exactly the export demand schedules (28). Conditions (69)–(70) are therefore equivalent to (28) given (68).

Balanced growth path implies static equilibrium. Consider a balanced growth path of the small open economy. The variable $Y_{h\$}$ does not exist on the balanced growth path, so we construct it: define

$$Y_{h\$} \equiv \sum_{i\in\mathcal{N}} p^Y_{hi} Y^{exp}_{hi},$$

and define $M_{h'j}$ as in (72). Substituting the export demand schedules (28) into (68) gives $F_{h\$}[\{Y^{exp}_{hi}\}] = \sum_i p^Y_{hi} Y^{exp}_{hi} = Y_{h\$}$, so the production function holds. Reversing the argument above, the export demand schedules imply that the quantities Y^{exp}_{hi} are cost-minimizing at output $Y_{h\$}$ and that the pricing condition (70) holds, which in turn implies (73) and hence $\Pi_{h\$} = S_{h\$}$, so the transfer (71) holds.

It remains to verify market clearing for dollars, (72), which states that trade is balanced. This follows from the household budget constraint. Substituting the transfer (9) with $\mathcal{T}_h = 0$, asset market clearing (15), the user cost equation (8), and the capital accumulation equation (6) into (2) gives

$$p_h^C C_h = \sum_{f\in\mathcal{F}} w_{hf} L_{hf} + \sum_{k\in\mathcal{K}} \left(R_{hk} K_{hk} - p_{hk}^X X_{hk}\right) + \sum_{i\in\mathcal{N}} (1 - 1/\mu_{hi}^Y) p_{hi}^Y Y_{hi}^Y.$$

Next, substitute the zero-profit conditions implied by (7): consumption spending equals the input cost of the consumption bundle, $p_{hi}^Y Y_{hi}^Y / \mu_{hi}^Y$ equals the input cost of industry i, and $p_{hk}^X X_{hk}$ equals the input cost of investment good k. Factor market clearing (13)–(14) cancels rental and wage payments, leaving

$$\sum_{h'\in\mathcal{H}} \sum_{j\in\mathcal{N}} p_{h'j}^Y \left[Y_{h,h'j}^C + \sum_{i\in\mathcal{N}} Y_{hi,h'j}^Y + \sum_{k\in\mathcal{K}} Y_{hk,h'j}^X \right] = \sum_{i\in\mathcal{N}} p_{hi}^Y Y_{hi}^Y.$$

Splitting the left-hand side into domestic ($h' = h$) and foreign ($h' \neq h$) purchases, and using the goods resource constraint (11), which for the small open economy reads $Y_{hj}^Y = Y_{h,hj}^C + \sum_i Y_{hi,hj}^Y + \sum_k Y_{hk,hj}^X + Y_{hj}^{exp}$, yields

$$\sum_{h'\neq h} \sum_{j\in\mathcal{N}} p_{h'j}^Y M_{h'j} = \sum_{j\in\mathcal{N}} p_{hj}^Y Y_{hj}^{exp} = Y_{h\$},$$

which is (72). All equations of the static system hold, so the solution set of the small open economy is contained in that of the static system. □

Proof of Proposition 9. Apply Proposition 8, treating Y_h^{exp} as the economy's only exported good. In the equivalent closed economy, the dollar industry therefore has a single input: its cost share on Y_h^{exp} is $1 - 1/\theta$, while its fixed-factor share is $1/\theta$. Both are constant.

All remaining input cost shares are also constant. This follows from the Cobb-Douglas assumptions for domestic technologies and the export aggregator; the internal composition of imported-input aggregates is immaterial because every imported good is produced linearly from dollars with productivity $1/p_{h'i'}^Y$. Thus, $\tilde{\Omega}_h^{soe}$ is invariant to a productivity shock. Since Φ^{soe} and $\boldsymbol{\mu}^{soe}$ are also invariant to the shock, the result follows from applying Proposition 4. □

D.4 General Export Demand System

Our results extend, with two modifications, to more general export demand systems. Suppose that exports satisfy

$$Y_{hi}^{exp} = \sum_{h'\neq h} \sum_{i'\in\mathcal{N}} D_{h'i',hi}(p_{h1}^Y, \ldots, p_{hN}^Y; \zeta_{h'i'}),$$

where $D_{h'i',hi}$ is the demand for good hi by user i' in country h' and the demand shifters $\zeta_{h'i'}$ do not depend on variables inside country h. Demand for each export good may thus depend flexibly on all domestic prices. We assume the resulting map from domestic prices to export quantities is invertible in a neighborhood of the initial balanced growth path, so that there exist functions $\tilde{P}_{hi}$ with

$$p^Y_{hi} = \tilde{P}_{hi}(Y^{exp}_{h1}, \ldots, Y^{exp}_{hN}).$$

Define the *revenue function*

$$\mathcal{R}_h(\mathbf{Y}^{exp}_h) = \sum_{i \in \mathcal{N}} Y^{exp}_{hi}\, \tilde{P}_{hi}(Y^{exp}_{h1}, \ldots, Y^{exp}_{hN}), \tag{74}$$

which maps export quantities to total export revenue. The revenue function can be represented as a constant-returns technology with a fixed factor through the standard construction $\tilde{\mathcal{R}}_h[\mathbf{Y}^{exp}_h, X] \equiv X\, \mathcal{R}_h(\mathbf{Y}^{exp}_h / X)$ with $X = 1$. The fixed factor earns the residual $\mathcal{R}_h - \sum_i Y^{exp}_{hi}\, \partial \mathcal{R}_h / \partial Y^{exp}_{hi}$.

Proposition 16. *The representation in Proposition 8 remains valid under the general export demand system with two modifications: (i) the dollar good is produced with the technology* (74)*; (ii) the output subsidy $\mu^Y_{h\$}$ is replaced by input-specific wedges*

$$\mu^{exp}_{hi} = \frac{1}{p^Y_{hi}} \frac{\partial \mathcal{R}_h}{\partial Y^{exp}_{hi}}.$$

The wedges in the proposition are the ratios of the export good's marginal revenue product to its price,

$$\mu^{exp}_{hi} = \frac{1}{p^Y_{hi}} \frac{\partial \mathcal{R}_h}{\partial Y^{exp}_{hi}} = 1 + \frac{1}{p^Y_{hi}} \sum_{j \in \mathcal{N}} Y^{exp}_{hj} \frac{\partial \tilde{P}_{hj}}{\partial Y^{exp}_{hi}}.$$

The dollar producer takes the export price p^Y_{hi} as given and does not internalize that additional exports depress it; the two marginal values differ by exactly the terms-of-trade term. Evaluated at the initial balanced growth path, μ^{exp}_{hi} is the input subsidy under which a price-taking producer facing effective input price $\mu^{exp}_{hi}\, p^Y_{hi}$ chooses the balanced-growth export quantities. The remainder of the proof of Proposition 8 applies directly.

Compared to the main text, the key difference is that the wedges are now specific to inputs, and in general they vary with the allocation. When own-price effects dominate, $\mu^{exp}_{hi} < 1$. Under the isoelastic system (28), $\tilde{P}_{hi} = D^{1/\theta}_{hi} (Y^{exp}_{hi})^{-1/\theta}$ implies $\partial \mathcal{R}_h / \partial Y^{exp}_{hi} =$

$(1-1/\theta)\, p^Y_{hi}$, so $\mu^{exp}_{hi} = 1 - 1/\theta$ for all goods, and we recover Proposition 8.

D.5 How to Apply Proposition 3 to a Small Open Economy

Applying Proposition 3 requires final expenditure shares Φ^{soe}_h, initial wedges $\boldsymbol{\mu}^{soe}_h$, an input cost share matrix $\tilde{\Omega}^{soe}_h$, and price elasticities σ^{soe}_h for the equivalent closed economy. Here, we supply these statistics. The equivalent closed static economy has the following industries:

- **Consumption goods:** A single consumption good h.
- **Intermediate inputs:** Intermediate input producers $i \in \mathcal{N}$.
- **Investment and capital goods:** Industries that produce investment goods, and their associated capital stocks, both indexed by $k \in \mathcal{K}$.
- **Imports:** Imports indexed by $i \in \mathcal{N}$ and $h' \in \mathcal{H}$ with $h' \neq h$, produced from dollars.
- **Dollar:** A dollar industry that buys export goods and produces dollars.
- **Labor endowments:** Labor inputs, indexed by labor type $f \in \mathcal{F}$.
- **Fixed factor endowment:** One fixed factor to capture the decreasing returns in the dollar sector.

The final expenditure share vector Φ^{soe} places weight one on the consumption good and zero on the remaining industries. The vector of initial wedges is given by:

$$\boldsymbol{\mu}^{soe}_h = [\underbrace{1}_{\text{Consumption}}, \underbrace{\boldsymbol{\mu}^Y_h}_{\text{Intermediates}}, \underbrace{\mathbf{1}_K}_{\text{Investment}}, \underbrace{\boldsymbol{\mu}^K_h}_{\text{Capital}}, \underbrace{\mathbf{1}_{(H-1)N}}_{\text{Imports}}, \underbrace{\left(1-\frac{1}{\theta}\right)}_{\text{Dollar}}, \underbrace{\mathbf{1}_F}_{\text{Labor}}, \underbrace{1}_{\text{Fixed factor}}].$$

The display below shows the input cost share matrix $\tilde{\Omega}^{soe}_h$, with rows giving the input cost shares of each industry, with industries in the same order as in $\boldsymbol{\mu}_h^{soe}$.

$$\tilde{\Omega}_h^{soe} = \begin{pmatrix} 0 & \tilde{\Omega}_{h,hN} & \mathbf{0}_{1,hX} & \mathbf{0}_{1,hK} & \tilde{\Omega}_{h,-hN} & 0 & \mathbf{0}_{1,hF} & 0 \\ \mathbf{0}_{hN,1} & \tilde{\Omega}_{hN,hN} & \mathbf{0}_{hN,hX} & \tilde{\Omega}_{hN,hK} & \tilde{\Omega}_{hN,-hN} & \mathbf{0}_{hN,\$} & \tilde{\Omega}_{hN,hF} & \mathbf{0}_{hN,1} \\ \mathbf{0}_{hX,1} & \tilde{\Omega}_{hX,hN} & \mathbf{0}_{hX,hX} & \mathbf{0}_{hX,hK} & \tilde{\Omega}_{hX,-hN} & \mathbf{0}_{hX,\$} & \mathbf{0}_{hX,hF} & \mathbf{0}_{hX,1} \\ \mathbf{0}_{hK,1} & \mathbf{0}_{hK,hN} & \mathbf{I}_{hK,hX} & \mathbf{0}_{hK,hK} & \mathbf{0}_{hK,-hN} & \mathbf{0}_{hK,\$} & \mathbf{0}_{hK,hF} & \mathbf{0}_{hK,1} \\ \mathbf{0}_{-hN,1} & \mathbf{0}_{-hN,hN} & \mathbf{0}_{-hN,hX} & \mathbf{0}_{-hN,hK} & \mathbf{0}_{-hN,-hN} & \mathbf{1}_{-hN,\$} & \mathbf{0}_{-hN,hF} & \mathbf{0}_{-hN,1} \\ 0 & (1-1/\theta)s_{hN}^{exp} & \mathbf{0}_{\$,hX} & \mathbf{0}_{\$,hK} & \mathbf{0}_{\$,-hN} & \mathbf{0}_{\$,\$} & \mathbf{0}_{\$,hF} & 1/\theta \\ \mathbf{0}_{F,1} & \mathbf{0}_{F,hN} & \mathbf{0}_{F,hX} & \mathbf{0}_{F,hK} & \mathbf{0}_{F,-hN} & \mathbf{0}_{F,\$} & \mathbf{0}_{F,hF} & 0 \\ 0 & \mathbf{0}_{1,hN} & \mathbf{0}_{1,hX} & \mathbf{0}_{1,hK} & \mathbf{0}_{1,-hN} & \mathbf{0}_{1,\$} & \mathbf{0}_{1,hF} & 0 \end{pmatrix}.$$

The first four rows are essentially identical to the rows associated with h in the world input cost matrix $\tilde{\Omega}$. More specifically, they give the input use of consumption goods, intermediates, investment and capital, including their use of imports, which are denoted by $-hN$. The fifth row states that imports get all their inputs from the dollar sector. The sixth row records the input cost shares of the dollar sector. The vector $[s_{hN}^{exp}]_{hi} = p_{hi}^Y Y_{hi}^{exp} / \sum_{i'} p_{hi'}^Y Y_{hi'}^{exp}$ is the share of h's exports coming from industry i. Since the dollar sector has a decreasing returns to scale technology, these shares are scaled by the returns to scale $(1-1/\theta)$ to obtain the input cost shares of intermediates, with the remaining share $1/\theta$ assigned to the fixed factor endowment in the last column. The last two rows are all zero, capturing that factor endowments do not use inputs.

The array of elasticities of substitution σ^{soe} is the same as in the world economy for consumption, intermediates, and investment. The imported goods, capital, and factor endowments have one or zero inputs so the elasticity of substitution is irrelevant for them. The only nontrivial case is the dollar sector, which operates a technology with decreasing returns to scale.

We use a fixed factor F_h to denote the quantity of the fixed factor that captures decreasing returns. The dollar sector technology is

$$Y_{h\$} = F_h^{\frac{1}{\theta}} M_h^{1-\frac{1}{\theta}}, \qquad M_h = \Big(\sum_{i \in \mathcal{N}} D_{hi}^{\frac{1}{\theta}} \left(Y_{hi}^{exp}\right)^{1-\frac{1}{\theta}} \Big)^{\frac{\theta}{\theta-1}},$$

which is a Cobb-Douglas aggregate of the fixed factor and a constant-elasticity aggregate of export goods with elasticity θ. The Allen-Uzawa elasticities are therefore

$$\sigma_{hi,hj}^{\$} = 1 + \theta \quad \text{for } i \neq j, \qquad \sigma_{hi,F_h}^{\$} = 1.$$

The objects above—Φ_h^{soe}, $\boldsymbol{\mu}_h^{soe}$, $\tilde{\Omega}_h^{soe}$, and σ_h^{soe}— are sufficient to apply Proposition 3, which yields all comparative statics for the small open economy.

E Appendix to Section 7: Richer Models of Asset Accumulation

In this appendix, we provide the explicit functional forms for $\mathcal{A}$ and $\mathcal{A}^{risk}$ and work through the derivation for Example 6.

E.1 Household Savings and Aggregate Asset Demand

To derive $\mathcal{A}$ and $\mathcal{A}^{risk}$, we provide the microfoundations for households' saving decisions. For the household block, decompose the common growth rate as $g = g_L + g_A$, where births grow at rate g_L and individual labor income grows at rate g_A. Only the household block uses this decomposition. Setting $g_L = 0$ and $g_A = g$ covers the case where all aggregate growth is labor-augmenting.

Preferences and budget. Individuals in country h die at Poisson rate ν_h and choose one type of domestic capital $k \in \mathcal{K}$ to operate at birth. A household born at date s and operating line k has preferences

$$V_{hk}(s) = \sup_{\{c_{hk}(t), a_{hk}(t), b_{hk}(t)\}} \mathbb{E}_s \int_s^\infty e^{-(\rho_h+\nu_h)(t-s)} \frac{c_{hk}(t)^{1-1/\gamma}}{1-1/\gamma}\, dt, \tag{75}$$

where γ is the intertemporal elasticity of substitution.

Let $n_{hk}(t)$ be financial net worth, $b_{hk}(t)$ holdings of the internationally traded risk-free bond, and $a_{hk}(t)$ the value of the household's claim on capital line hk, so that $n_{hk}(t) = b_{hk}(t) + a_{hk}(t)$. The bond earns $r_0(t)$. The capital claim has pre-tax expected return $r_{hk}(t)$, carries an idiosyncratic Brownian shock with standard deviation φ_{hk}, and is taxed at the additive wealth-tax rate τ^K_{hk}. Following Angeletos (2007), individuals cannot diversify the idiosyncratic capital risk they bear. Let $y_h(t)$ denote labor income plus lump-sum transfers received while alive. Financial net worth follows

$$dn_{hk}(t) = \Big[y_h(t) + \nu_h n_{hk}(t) + r_0(t) n_{hk}(t) + (r_{hk}(t) - \tau^K_{hk} - r_0(t)) a_{hk}(t) - p^C_h(t) c_{hk}(t)\Big] dt + \varphi_{hk} a_{hk}(t)\, dZ_{hk}(t). \tag{76}$$

The term $\nu_h n_{hk}(t)$ is the mortality bonus from an actuarially fair annuity as in Blanchard (1985). Annuity providers transfer the wealth of the deceased to survivors and earn zero profits. We impose the natural borrowing limit: households have to be able to repay in all states of the world, which means that bonds cannot be more negative than the net present value of future labor and transfer income. We also assume that financial wealth satisfies $n_{hk}(s) = 0$ at birth. The capital income shocks $dZ_{hk}(t)$ are specific to households and wash out in the aggregate.

Consider a candidate balanced growth path along which r_0, r_{hk}, p_h^C are constant, while individual non-asset income grows at rate g_A. Write $\widehat{n}_{hk}(t) = e^{-g_A t} n_{hk}(t)$, $\widehat{c}_{hk}(t) = e^{-g_A t} c_{hk}(t)$, and $\widehat{y}_h = e^{-g_A t} y_h(t)$. Define detrended effective wealth (sum of financial and human wealth) and the risky share of effective wealth to be:

$$\widehat{\omega}_{hk}(t) \equiv \widehat{n}_{hk}(t) + \frac{\widehat{y}_h}{r_0 + \nu_h - g_A}, \qquad \zeta_{hk}(t) \equiv \frac{e^{-g_A t} a_{hk}(t)}{\widehat{\omega}_{hk}(t)}, \tag{77}$$

where $r_0 + \nu_h > g_A$ so that human wealth is finite. Effective wealth evolves as

$$d\widehat{\omega}_{hk}(t) = \Big[(r_0 + \nu_h - g_A + \zeta_{hk}(t)(r_{hk}(t) - \tau_{hk}^K - r_0(t)))\widehat{\omega}_{hk}(t) - p_h^C \widehat{c}_{hk}(t)\Big] dt + \varphi_{hk}\zeta_{hk}(t)\widehat{\omega}_{hk}(t)\, dZ_{hk}(t). \tag{78}$$

This is isomorphic to a Merton (1969) portfolio problem. At an interior optimum, the Merton portfolio shares are

$$\zeta_{hk} = \gamma \frac{r_{hk} - \tau_{hk}^K - r_0}{\varphi_{hk}^2} = \gamma \frac{S_{hk}}{\varphi_{hk}}, \qquad S_{hk} \equiv \frac{r_{hk} - \tau_{hk}^K - r_0}{\varphi_{hk}}, \tag{79}$$

where S_{hk} is the Sharpe ratio. Consumption is proportional to effective wealth:

$$p_h^C \widehat{c}_{hk}(t) = \xi_{hk}\widehat{\omega}_{hk}(t), \qquad \xi_{hk} = \gamma \left[\rho_h + \nu_h - \left(1 - \frac{1}{\gamma}\right)\left(r_0 + \nu_h + \frac{\gamma}{2} S_{hk}^2\right)\right].$$

We assume $\xi_{hk} > 0$.

Let $g_{\omega,hk}$ be the (non-detrended) growth rate of the expected level of effective wealth conditional on survival

$$g_{\omega,hk} = g_A + \frac{\mathbb{E}[d\widehat{\omega}_{hk}(t)/dt]}{\mathbb{E}\widehat{\omega}_{hk}(t)}.$$

It satisfies

$$g_{\omega,hk} = [r_0 + \nu_h] + \zeta_{hk}[r_{hk} - \tau_{hk}^K - r_0] - \frac{p_h^C \widehat{c}_{hk}}{\widehat{\omega}_{hk}},$$

which, after substitution, is

$$g_{\omega,hk} = \gamma(r_0 - \rho_h) + \frac{\gamma(\gamma+1)}{2} S_{hk}^2. \tag{80}$$

Since the choice of capital line affects the value function only through the squared Sharpe ratio S_{hk}^2, and since all capital lines are active, indifference across capital lines within a country implies that all capital lines deliver the same Sharpe ratio S_h:

$$\frac{r_{hk} - \tau_{hk}^K - r_0}{\varphi_{hk}} = S_h, \qquad k \in \mathcal{K}, \tag{81}$$

which is (32) in the main text. This implies that the growth of effective wealth conditional on survival $g_{\omega,hk}$ and the consumption share of effective wealth ξ_{hk} do not depend on k and only vary as a function of country h. Hence, going forward, we drop the k subscript.

We derive the aggregate demand for bonds and risky assets in each country by aggregating over all households in that country. Let $N_{h0}e^{g_L s}ds$ be the flow of births in country h at date s, and $\eta_{hk}(s)$ the share of newborns choosing line k. For a household variable x, write $x_{hk}(s,t)$ for the process of a household born at s operating line k, zero before birth and after death. Aggregates are

$$B_{hk}(t) = \int_{-\infty}^{t} \eta_{hk}(s)\, \mathbb{E}[a_{hk}(s,t)]\, N_{h0}e^{g_L s} ds, \tag{82}$$

$$B_{h0}(t) = \int_{-\infty}^{t} \sum_k \eta_{hk}(s)\, \mathbb{E}[b_{hk}(s,t)]\, N_{h0}e^{g_L s} ds, \tag{83}$$

$$C_h(t) = \int_{-\infty}^{t} \sum_k \eta_{hk}(s)\, \mathbb{E}[c_{hk}(s,t)]\, N_{h0}e^{g_L s} ds. \tag{84}$$

On a balanced growth path the entry shares η_{hk} are constant and every line with $\eta_{hk} > 0$ delivers the same value to a newborn. Define aggregate human wealth—the capitalized value of non-asset income—by

$$\mathbb{H}_h \equiv \frac{\sum_{f\in\mathcal{F}} w_{hf} L_{hf} + T_h}{r_0 + \nu_h - g_A}, \tag{85}$$

where T_h collects the rebates defined in (9). This expression uses that nonasset income per household grows at rate g_A, while households die at rate ν_h. We assume $r_0 + \nu_h > g_A$, so that human wealth is finite.

We next aggregate effective wealth across cohorts. Let a denote age. Since births grow at rate g_L and households die at rate ν_h, the cross-sectional age density among surviving households on the balanced growth path is

$$f_h(a) = (\nu_h + g_L)e^{-(\nu_h+g_L)a}, \qquad a \geq 0.$$

Relative to a newborn at date t, a household of age a was born when wages, and hence its initial human wealth, were lower by the factor $e^{-g_A a}$. Conditional on survival, its expected effective wealth has subsequently grown at rate $g_{\omega,h}$. Therefore, its expected effective wealth at date t, relative to the human wealth of a newborn at date t, is

$$e^{(g_{\omega,h}-g_A)a}.$$

Since all households alive at time t have the same human wealth, and newborn households only have human wealth, it follows that aggregate effective wealth is

$$\begin{aligned} W_h &= \mathbb{H}_h \int_0^\infty (\nu_h + g_L)e^{-(\nu_h+g_L)a} e^{(g_{\omega,h}-g_A)a}\, da \\ &= \chi_h \mathbb{H}_h, \end{aligned} \tag{86}$$

where

$$\chi_h \equiv \frac{\nu_h + g_L}{\nu_h + g_L + g_A - g_{\omega,h}} = \frac{\nu_h + g_L}{\nu_h + g - g_{\omega,h}}, \qquad g \equiv g_L + g_A. \tag{87}$$

The condition

$$\nu_h + g > g_{\omega,h}$$

ensures that aggregate effective wealth is finite: otherwise, the effective wealth of sufficiently old surviving cohorts grows faster with age than their mass declines.

Financial wealth is effective wealth less human wealth, and (79) makes after-tax excess-return income equal to γS_h^2 times effective wealth:

$$B_{h0} + \sum_{k\in\mathcal{K}} B_{hk} = (\chi_h - 1)\mathbb{H}_h, \tag{88}$$

$$\sum_{k\in\mathcal{K}} \left(r_{hk} - \tau_{hk}^K - r_0\right) B_{hk} = \gamma S_h^2 \chi_h \mathbb{H}_h. \tag{89}$$

For the set of returns satisfying (81), the asset-demand functions in (30)–(31) are

$$\mathcal{A}_h(r_0, S_h) = \frac{g_{\omega,h} - g_A}{(r_0 + \nu_h - g_A)(\nu_h + g - g_{\omega,h})}, \tag{90}$$

$$\mathcal{A}_h^{\text{risk}}(r_0, S_h) = \frac{\gamma S_h^2(\nu_h + g_L)}{(r_0 + \nu_h - g_A)(\nu_h + g - g_{\omega,h})}. \tag{91}$$

The first is financial wealth relative to non-asset income. The second is after-tax risky excess-return income relative to non-asset income; its numerator is the left-hand side of (89).

E.2 Proofs for this Section

Proof of Proposition 10. Consider a balanced growth path satisfying (3)–(15) and (30)–(33).

Step 1: rewrite the household budget. Capital taxes are rebated, so after-tax capital income $(r_{hk} - \tau^K_{hk} - g)B_{hk}$ plus the rebate $\tau^K_{hk} B_{hk}$ equals $(r_{hk} - g)B_{hk}$. Together with the exogenous transfer $\mathcal{T}_h$, we obtain

$$\begin{aligned} p^C_h C_h = & \sum_f w_{hf} L_{hf} + \sum_i \left(1 - \frac{1}{\mu^Y_{hi}}\right) p^Y_{hi} Y^Y_{hi} \\ & + \sum_k (r_{hk} - g) B_{hk} + (r_0 - g) B_{h0} + \mathcal{T}_h. \end{aligned} \tag{92}$$

Step 2: convert capital income into wedge profits. Asset-market clearing $B_{hk} = p^X_{hk} K_{hk}$, the user cost $R_{hk} = (r_{hk} + \delta_k) p^X_{hk}$, and balanced-growth investment $X_{hk} = (g + \delta_k) K_{hk}$ give

$$(r_{hk} - g) B_{hk} = R_{hk} K_{hk} - p^X_{hk} X_{hk} = \left(1 - \frac{1}{\mu^K_{hk}}\right) R_{hk} K_{hk}, \qquad \mu^K_{hk} = \frac{r_{hk} + \delta_k}{g + \delta_k}. \tag{93}$$

Step 3: match to the static economy. With (93), equation (92) is the household budget of the equivalent static economy in Proposition 2, with the transfer given by $\tilde{\mathcal{T}}_h = \mathcal{T}_h + (r_0 - g) B_{h0}$. Production functions, pricing equations, and resource constraints are unchanged. Every balanced growth path is therefore an equilibrium of that static economy.

□

E.3 Details on Example 6

Asset-market clearing is given by

$$\frac{p^X K}{wL} = \mathcal{A}(r), \tag{94}$$

and the CES technology of Example 2 gives

$$\frac{p^X K}{wL} = \frac{\alpha}{1-\alpha} \left(\frac{p}{w}\right)^{1-\sigma_{KL}} (r + \delta)^{-\sigma_{KL}}. \tag{95}$$

Shephard's lemma and the unit-cost equation imply

$$d\log\frac{p}{w} = \frac{1}{1-\alpha}\left[-d\log A + \frac{\alpha}{r+\delta}dr\right], \tag{96}$$

so differentiating (95) yields

$$d\log\frac{p^X K}{wL} = \frac{\sigma_{KL}-1}{1-\alpha}d\log A - \frac{\sigma_{KL}-\alpha}{1-\alpha}\frac{dr}{r+\delta}. \tag{97}$$

Setting this equal to $[\mathcal{A}'(r)/\mathcal{A}(r)]dr$ gives the return response reported in Example 6:

$$\frac{dr}{d\log A} = \frac{(\sigma_{KL}-1)/(1-\alpha)}{\mathcal{A}'(r)/\mathcal{A}(r) + [(\sigma_{KL}-\alpha)/(1-\alpha)](r+\delta)^{-1}}. \tag{98}$$

Let

$$\pi_K \equiv \frac{RK - p^X X}{p^C C} = \lambda_K\left(1-\frac{1}{\mu^K}\right) \tag{99}$$

be the capital-sector profit share of the equivalent static economy. Holding the return fixed, Example 2 gives $d\log C^{\text{exog. returns}}/d\log A$. A change in the return moves the capital wedge by $d\log\mu^K = dr/(r+\delta)$ and the capital stock by $\partial\log K/\partial\log\mu^K = -\sigma_{KL}/(1-\alpha)$. Hence

$$\frac{d\log C}{d\log A} = \frac{d\log C^{\text{exog. returns}}}{d\log A} - \pi_K\frac{\sigma_{KL}}{1-\alpha}\frac{1}{r+\delta}\frac{dr}{d\log A}. \tag{100}$$

Endogenous returns dampen the fixed-return reallocation effect. If $\sigma_{KL} > 1$, the productivity improvement raises capital demand and the return, which limits accumulation; if $\sigma_{KL} < 1$, the return falls and limits the contraction. At the Golden Rule $\pi_K = 0$ and the return channel vanishes.

The log-utility specialization of Appendix E.1 has $g_L = g_A = 0$, $\gamma = 1$, and no idiosyncratic risk. Its asset-demand schedule and elasticity are

$$\mathcal{A}(r) = \frac{r-\rho_h}{(r+\nu_h)(\nu_h+\rho_h-r)}, \qquad \frac{\mathcal{A}'(r)}{\mathcal{A}(r)} = \frac{1}{r-\rho_h} - \frac{1}{r+\nu_h} + \frac{1}{\nu_h+\rho_h-r}. \tag{101}$$

As mortality vanishes, the schedule becomes arbitrarily elastic around the representative-agent return $r = \rho_h$, and (98) converges to the fixed-return case.

F Appendix to Section 8: Quantitative Model

F.1 Nonlinear Solution

This section describes how we compute the effects of large shocks. The logic is as follows. Proposition 3 expresses the first-order response of every balanced growth path outcome to productivity, wedge, and capital-tax shocks as a function of the sufficient statistics $(\boldsymbol{\Phi}, \tilde{\Omega}, \boldsymbol{\mu}, \boldsymbol{\sigma})$. The sufficient statistics are themselves equilibrium outcomes covered by the proposition. As a result, they satisfy an ordinary differential equation along any path of shocks, with a right-hand side that is a known function of their current values. Integrating this differential equation yields the global change in the sufficient statistics; integrating a second time along the solution path yields the global change in any other outcome.

Shock path. We index shocks by $t \in [0,1]$ and connect initial and terminal parameters by the linear path

$$\boldsymbol{\zeta}(t) \;\equiv\; \big(\log \boldsymbol{A}(t), \log \boldsymbol{\mu}^{Y}(t), \boldsymbol{\tau}^{K}(t)\big) \;=\; (1-t)\,\boldsymbol{\zeta}(0) + t\,\boldsymbol{\zeta}(1). \tag{102}$$

Each t indexes a balanced growth path, and $t = 1$ corresponds to the new balanced growth path after the full shock. Since balanced growth path outcomes depend only on parameter levels, the terminal point does not depend on the choice of path; the linear path is a computational convenience.

The sufficient statistics follow a differential equation. Let $\mathcal{S}(t) \equiv \big(\boldsymbol{\Phi}(t), \tilde{\Omega}(t), \boldsymbol{\mu}(t), \boldsymbol{\sigma}(t)\big)$ collect the sufficient statistics on the balanced growth path indexed by t. Each component has a known derivative with respect to t. Changes in expenditure shares $\boldsymbol{\Phi}$ and cost shares $\tilde{\Omega}$ are part of the equation system in Proposition 3: equation (23) determines $d\boldsymbol{\Phi}$ and equation (21) determines $d\log\tilde{\Omega}$, in both cases as functions of $\mathcal{S}(t)$ and the shock increment. Changes in wedges are mechanical: wedges on intermediate goods $\boldsymbol{\mu}^{Y}$ move one-for-one with the shock, and capital wedges satisfy $d\log\mu^{K}_{hk} = d\tau^{K}_{hk}/(\delta_k + r_{hk}(t))$ with $r_{hk}(t) = \rho + g/\gamma + \tau^{K}_{hk}(t)$. Finally, in our nested CES system, the Allen–Uzawa elasticities are functions only of the nest elasticities $\boldsymbol{\theta}$ and the current cost shares, $\boldsymbol{\sigma}(t) = \Sigma\big(\tilde{\Omega}(t); \boldsymbol{\theta}\big)$, so changes in $\boldsymbol{\sigma}$ follow from changes in $\tilde{\Omega}$. Stacking these derivatives, the sufficient statistics solve the initial value problem

$$\frac{d\mathcal{S}}{dt} = H\big(\mathcal{S}(t), \boldsymbol{\zeta}'(t); \boldsymbol{\theta}\big), \qquad \mathcal{S}(0) \text{ given,} \tag{103}$$

where the function H collects the equations above and $\mathcal{S}(0)$ contains the calibrated statistics on the initial balanced growth path. Because the system in Proposition 3 is linear in shocks, H is linear in $\boldsymbol{\zeta}'(t)$.

Global changes in other outcomes. Consider any other outcome Z, such as a price, a quantity, a sales share, or consumption in a given country. Proposition 3 expresses its first-order response as a function of the sufficient statistics and the shock increment,

$$\frac{d\log Z}{dt} = H_Z\big(\mathcal{S}(t), \boldsymbol{\zeta}'(t); \boldsymbol{\theta}\big). \tag{104}$$

Given the solution $\mathcal{S}(t)$ to (103), the global change in Z is the integral of its local responses along the path,

$$\log Z(1) - \log Z(0) = \int_0^1 H_Z\big(\mathcal{S}(t), \boldsymbol{\zeta}'(t); \boldsymbol{\theta}\big)\, dt. \tag{105}$$

Implementation. We solve (103) and (104) jointly with a forward Euler scheme. We partition $[0, 1]$ into N_{steps} steps of length $\Delta t = 1/N_{\text{steps}}$. At each step, we solve the linear system in Proposition 3 at the current statistics $\mathcal{S}(t)$ for the shock increment $\boldsymbol{\zeta}'(t)\Delta t$, update the expenditure shares, cost shares, wedges, and elasticities, and cumulate the changes in outcomes. The cumulated changes converge to the solution of the differential equation as $\Delta t \to 0$; Baqaee and Farhi (2024) use the same approach for static economies.

The procedure shows that the information required for global comparative statics is the same as for local comparative statics: the expenditure shares, cost shares, and wedges on the initial balanced growth path, together with the nest elasticities $\boldsymbol{\theta}$. In particular, there is no need to recover the level share parameters of the nested CES system, in the spirit of exact hat algebra (Dekle et al., 2008).

F.2 Calibration of benchmark model

Below we describe how we set the inputs to our benchmark models: initial consumption expenditure shares Φ, the input cost share matrix $\tilde{\Omega}$, the initial wedges μ, the array of elasticities $\boldsymbol{\sigma}$, and the transfers $\mathcal{T}_h$ needed to rationalize observed trade deficits.

For the benchmark model, we use two primary data sources. First, we use the World Input-Output Database (WIOD) and the associated Socio-Economic Accounts for consumption spending, intermediate input use, and labor inputs (Timmer et al., 2015). Second, we use the investment flow data from Ding (2022) for investment spending. To ensure consistency, we aggregate WIOD sectors to match the sectoral classification in Ding (2022), yielding 27 sectors. We use data from 1995–2009 for WIOD, and 1997 for investment flows.

Consumption share by country Φ. We construct consumption shares by country as a time-average of consumption shares in the WIOD. Specifically, for each period in the calibration sample, we define country h's consumption as final expenditure net of investment summed across all goods. Writing $CONS_{ht}$ for this quantity, we define Φ_h for country h by dividing this number by total world consumption, averaged over all years:

$$\Phi_h = \text{mean}_t \left(\frac{CONS_{ht}}{\sum_{h'} CONS_{h't}} \right) \qquad h \in \mathcal{H}.$$

Input cost matrix $\tilde{\Omega}$. From Section 4.2, we have that $\tilde{\Omega}$ has the following structure.

$$\tilde{\Omega} = \begin{pmatrix} \mathbf{0}_{H,H} & \tilde{\Omega}_{H,HN} & \mathbf{0}_{H,HX} & \mathbf{0}_{H,HK} & \mathbf{0}_{H,HF} \\ \mathbf{0}_{HN,H} & \tilde{\Omega}_{HN,HN} & \mathbf{0}_{HN,HX} & \tilde{\Omega}_{HN,HK} & \tilde{\Omega}_{HN,HF} \\ \mathbf{0}_{HX,H} & \tilde{\Omega}_{HX,HN} & \mathbf{0}_{HX,HX} & \mathbf{0}_{HX,HK} & \mathbf{0}_{HX,HF} \\ \mathbf{0}_{HK,H} & \mathbf{0}_{HK,HN} & \mathbf{I}_{HK,HX} & \mathbf{0}_{HK,HK} & \mathbf{0}_{HK,HF} \\ \mathbf{0}_{HF,H} & \mathbf{0}_{HF,HN} & \mathbf{0}_{HF,HX} & \mathbf{0}_{HF,HK} & \mathbf{0}_{HF,HF} \end{pmatrix}. \tag{106}$$

We populate the submatrices as follows; all elements not listed are zero. Let $k(i) \in \mathcal{K}$ denote the capital good used by industry i, and let mean_t denote the average over $t \in \{1995, \ldots, 2009\}$.

$$\begin{aligned}
[\tilde{\Omega}_{H,HN}]_{h,h'i'} &= \text{mean}_t \left(\frac{\text{CONS}_{h,h'i',t}}{\text{CONS}_{h,t}} \right), && h, h' \in \mathcal{H},\ i' \in \mathcal{N}, \\
[\tilde{\Omega}_{HN,HN}]_{hi,h'i'} &= \text{mean}_t \left(\frac{\text{INT}_{hi,h'i',t}}{\text{TC}_{hi,t}} \right), && h, h' \in \mathcal{H},\ i, i' \in \mathcal{N}, \\
[\tilde{\Omega}_{HN,HK}]_{hi,hk(i)} &= \text{mean}_t \left(\frac{\text{CAP}_{hi,t}}{\text{TC}_{hi,t}} \right), && h \in \mathcal{H},\ i \in \mathcal{N}, \\
[\tilde{\Omega}_{HN,HF}]_{hi,hf} &= \text{mean}_t \left(\frac{\text{COMP}_{hi,hf,t}}{\text{TC}_{hi,t}} \right), && h \in \mathcal{H},\ i \in \mathcal{N},\ f \in \mathcal{F}, \\
[\tilde{\Omega}_{HX,HN}]_{hk(i),h'i'} &= \frac{\text{GFCF}_{hi,h'i',1997}}{\text{GFCF}_{hi,1997}}, && h, h' \in \mathcal{H},\ i' \in \mathcal{N}, \\
[\mathbf{I}_{HK,HX}]_{hk,hk} &= 1, && h \in \mathcal{H},\ k \in \mathcal{K},
\end{aligned}$$

where $\text{TC}_{hi,t} \equiv \text{INT}_{hi,t} + \text{CAP}_{hi,t} + \text{COMP}_{hi,t}$ and dropping a source index denotes summation over it. All values are nominal. $\text{CONS}_{h,h'i',t}$ is spending by country h on consumption goods from industry i' in country h'; $\text{INT}_{hi,h'i',t}$, $\text{CAP}_{hi,t}$, and $\text{COMP}_{hi,hf,t}$ are the intermediate expenditure, capital compensation, and labor compensation of industry

i in country h, with capital and labor sourced domestically. These come from the WIOD and its Socio-Economic Accounts. The only exception to the time-averaging is the composition of investment costs for each destination industry-country pair across alternative industry-country origins. This share comes from the investment flow matrix of Ding (2022), which is available only for 1997.

For the investment cost shares, we use the investment flow table from Ding (2022) which creates estimates for the share of investment in an industry that is sourced from different countries and industries. The data is only available for 1997, where $GFCF_{hi,h'i',1997}$ is Ding's estimate of the nominal investment expenditure of industry i on investment goods from industry i' in country h', and $GFCF_{hi,1997} = \sum_{h',i'} GFCF_{hi,h'i',1997}$ is the total investment expenditure of industry i.

Initial wedges μ. For the initial wedges, we set the capital wedge associated with an industry's capital good equal to the ratio between capital compensation and investment:

$$\mu^K_{hk(i)} = \text{mean}_{t\in\{1995,\ldots,2009\}}\left(\frac{CAP_{hi,t}}{GFCF_{hk(i),t}}\right),$$

reflecting that $CAP_{hi,t}$ is the revenue associated with the capital good, and $GFCF_{hk(i),t}$ its cost of production. For intermediate inputs, the wedges $\boldsymbol{\mu}^Y$ are set equal to one, which is consistent with WIOD that have costs being equal to revenue. We set all other initial wedges to one. That is, with some abuse of notation, $\boldsymbol{\mu}^C = \boldsymbol{\mu}^X = \boldsymbol{\mu}^L = 1$. We truncate μ^K_{hk} to 1, setting $\mu^K_{hk} = 1$ when the formula above implies $\mu^K_{hk} < 1$.

Array of elasticities of substitution $\boldsymbol{\sigma}$. The production structure with Cobb-Douglas aggregation of Armington nests means that all elasticities of substitution are determined by the Armington elasticity θ and the cost shares $\tilde{\Omega}$ constructed above. We set $\theta = 5$ in the baseline, corresponding to a trade elasticity of 4, in line with, e.g., Simonovska and Waugh (2014).

Given θ, the array $\boldsymbol{\sigma}$ is defined as follows. Let $\tilde{\Omega}^u_{a,i} = \sum_{h'}[\tilde{\Omega}]_{a,h'i}$, $u \in \{C, Y, X\}$, denote the Cobb–Douglas cost share that producer a places on the industry-i nest. For consumption good h, the elasticity between distinct inputs $h'i$ and $h''j$ is

$$(\sigma^C)^h_{h'i,h''j} = \begin{cases} 1 + \dfrac{\theta - 1}{\tilde{\Omega}^C_{h,i}}, & i = j,\ h' \neq h'', \\ 1, & \text{otherwise.} \end{cases}$$

For intermediate producer hi, with b and c any two distinct inputs,

$$(\sigma^Y)^{hi}_{b,c} = \begin{cases} 1 + \dfrac{\theta - 1}{\tilde{\Omega}^Y_{hi,j}}, & b = h'j,\ c = h''j,\ h' \neq h'', \\ 1, & \text{otherwise.} \end{cases}$$

For investment-good producer hk, the elasticity between distinct inputs $h'i$ and $h''j$ is

$$(\sigma^X)^{hk}_{h'i,h''j} = \begin{cases} 1 + \dfrac{\theta - 1}{\tilde{\Omega}^X_{hk,i}}, & i = j,\ h' \neq h'', \\ 1, & \text{otherwise.} \end{cases}$$

The "otherwise" case consists of pairs of inputs drawn from different industry nests as well as intermediate–capital, intermediate–labor, capital–labor, and pairs of different labor types f. Each intermediate producer uses a single associated capital good, so there is no capital–capital pair. Capital-service producers have one input and factors have none, so they have no pairs of non-zero inputs for which elasticities can be defined.

Transfers. Last, we note that given the other objects, the exogenous transfers can be backed out residually. Recall that given the other objects, it is possible to express the sales of each industry relative to total world consumption as the vector that solves the equation

$$\lambda_a = \Phi_a + \sum_b \frac{1}{\mu_b} \lambda_b \tilde{\Omega}_{b,a}. \tag{107}$$

Given λ, we can define the transfer as consumption net of income:

$$\mathcal{T}_h = \Phi_h - \left[\sum_f \lambda_{hf} + \sum_{i \in \mathcal{N}} \lambda_{hi}(1 - 1/\mu_i) + \sum_{k \in \mathcal{K}} (1 - 1/\mu_{hk}) \lambda_{hk} \right].$$

The expression in brackets equals the country's total consumption plus net exports, so the transfer equals a country's trade deficit.

F.3 Calibration in extension with endogenous returns

Table 7 lists the additional parameters needed to solve the model with endogenous returns. The net foreign asset positions are from the External Wealth of Nations Database (Lane and Milesi-Ferretti, 2018), and we write b_{h0} for their value normalized by world consumption. The risk-free rate, population growth, and the technology growth rate are based on long-term yields on US treasuries, GDP-weighted working age population growth rates, and per capita growth in US real GDP (UN Population Prospects, FRED, and the Penn World Table). Our results are not very sensitive to the precise values of the interest rate and these growth rates. We use industry-specific depreciation rates from the Bureau of Economic Analysis, which given capital wedges μ_{hk} and the growth rate $g = g_A + g_L$, let us back out r_{hk} for each capital good k in country h.

On the balanced-growth path, a position b_{h0} generates net foreign-asset income $(r_0 - g)b_{h0}$. Exogenous transfers are set to

$$\mathcal{T}_h^{\text{fixed}} = \Phi_h - \left[\sum_f \lambda_{L_{hf}} + \sum_{k \in \mathcal{K}} (1 - 1/\mu_{hk}^K) \lambda_{K_{hk}} + (r_0 - g) b_{h0} \right].$$

This transfer is rebated lump-sum to households in the country, and thus acts as an extra source of non-asset income. In the data, the size of the transfer corresponds to the size of the trade deficit net of foreign asset income $(r_0 - g)b_{h0}$.

Table 7: Calibration targets

Variable	**Description**	**Value and source**
b_{h0}	Net foreign asset positions	Lane and Milesi-Ferretti (2018)
r_0	Risk-free rate	0.025 (FRED)
g_L	Population growth rate	0.004 (UN WPP)
g_A	Labor-augmenting technology	0.02 (Penn World Table)
δ_k	Depreciation by industry	BEA
γ	Intertemporal elasticity of sub.	0.5
ρ_h	Impatience parameter	See text.
ν_h	Mortality rate	See text.
S_h	Sharpe ratio	See text.

The remaining parameters in Table 7 are needed to pin down elasticities of the asset demand block. We set the elasticity of intertemporal substitution $\gamma = 0.5$. For the remaining household savings parameters, we calibrate the discount rate, mortality, and Sharpe ratios, (ρ_h, ν_h, S_h). To do so, we use three targets: (1) financial wealth relative to nonasset income, $\left[b_{h0} + \sum_{k \in \mathcal{K}} \frac{\lambda_{K_{hk}}}{r_{hk} + \delta_k} \right] / \left[\sum_{f \in \mathcal{F}_h} \lambda_{L_{hf}} + \mathcal{T}_h^{\text{fixed}} \right]$, where we have expressed everything relative to world consumption, using that the value of each capital stock can be expressed in terms of the revenue share from the input-output table and previously derived returns r_{hk} and the value of capital $p_{hk}^X K_{hk} = \frac{\lambda_{K_{hk}}}{r_{hk} + \delta_k}$; (2) a semi-elasticity of total wealth relative to nonasset income with respect to the risk-free rate (in each country) of 18 (based on Auclert et al., 2021); (3) the total excess returns from risky assets relative to nonasset income $\left(\sum_{k\in\mathcal{K}}\left[r_{hk}-r_0\right]\lambda_{K_{hk}}/(r_{hk}+\delta_k)\right)/\left[\sum_{f\in\mathcal{F}}\lambda_{L_{hf}}+\mathcal{T}_h^{fixed}\right]$.

Using these targets, we solve for the mortality rate ν_h, the Sharpe ratio S_h, and the growth rate of wealth conditional on survival, $g_{\omega,h}$. Given this, we can back out the subjective discount rate as

$$\rho_h = r_0 + \frac{\gamma+1}{2}S_h^2 - \frac{g_{\omega,h}}{\gamma}.$$

F.4 Construction of $\tilde{\Omega}^{soe}$

Appendix D.5 gives the block structure of $\tilde{\Omega}^{soe}$; Appendix F.2 explains how $\tilde{\Omega}^{soe}$ is constructed in our quantitative model. Three steps are specific to the data.

First, the construction retains every world intermediate-good variety rather than aggregating imports into a composite. Home consumption, production, and capital rows are copied unchanged from the world matrix, so the origin composition of home imports is carried over. Only the foreign rows are replaced, with all their inputs coming from the dollar sector.

Second, the investment and capital blocks of Appendix D.5 collapse into a single home block, copied from the home capital-good rows of the calibrated world matrix. That is, instead of having a capital good produced by a unique investment good which in turn is produced by intermediate inputs, we have the capital good being produced from intermediates directly. Since we only remove an identity matrix, all results are unaffected.

Third, Appendix D.5 places weight one on home consumption and zero elsewhere, which presumes balanced trade. In the data $NX_h \neq 0$, so we normalize home consumption to one and place the calibrated surplus at the dollar node,

$$\Phi^{soe} = e_{C_h} + \frac{NX_h}{p_h^C C_h}\, e_{\$},$$

where e_a is the unit vector at node a. A surplus enters as positive external demand for dollars, a deficit as negative demand. This is the $\mathcal{T}_h \neq 0$ case set aside in Appendix D.3, and the external balance enters here and only here.

Cost weights are then $\tilde{\lambda}^{soe} = \Phi^{soe}(I-\tilde{\Omega}^{soe})^{-1}$, the object of Proposition 9. Apart from the trade elasticity, the construction requires home consumption and input shares by origin, home investment flows, home export shares, the home trade balance, and the calibrated wedges on intermediate goods. Foreign production networks are not needed.

F.5 Derivation of Equation (34)

Consider first-order changes around the benchmark with no initial intermediate-good wedges, $\mu^Y_{hi} = 1$. Productivities and factor endowments are held fixed, and trade is balanced before and after the perturbation.

Define permanent domestic product as GDP net of investment:

$$PDP_h \equiv \sum_{j \in \mathcal{N}} p^Y_{hj} Y^Y_{hj} - \sum_{\substack{j \in \mathcal{N} \\ h' \in \mathcal{H},\, i \in \mathcal{N}}} p^Y_{h'i} Y^Y_{hj,h'i} - \sum_{k \in \mathcal{K}} p^X_{hk} X_{hk}.$$

For each origin–industry pair $h'i$, define its contribution to country h's net permanent output by

$$q_{h,h'i} \equiv \mathbf{1}\{h' = h\} Y^Y_{h'i} - \sum_{j \in \mathcal{N}} Y^Y_{hj,h'i} - \sum_{k \in \mathcal{K}} Y^X_{hk,h'i}.$$

Zero profits in investment-good production imply

$$PDP_h = \sum_{\substack{h' \in \mathcal{H} \\ i \in \mathcal{N}}} p^Y_{h'i} q_{h,h'i}.$$

Define country h's net exports of good $h'i$ by

$$NX_{h,h'i} \equiv p^Y_{h'i} \left(q_{h,h'i} - Y^C_{h,h'i} \right).$$

Thus, $NX_{h,hi}$ is the value of exports of domestic good hi, whereas $NX_{h,h'i}$ is minus the value of imports when $h' \neq h$. Since

$$p^C_h C_h = \sum_{\substack{h' \in \mathcal{H} \\ i \in \mathcal{N}}} p^Y_{h'i} Y^C_{h,h'i},$$

we have

$$PDP_h = p^C_h C_h + \sum_{\substack{h' \in \mathcal{H} \\ i \in \mathcal{N}}} NX_{h,h'i}.$$

Balanced trade implies that the final term is zero throughout the perturbation, so $PDP_h = p^C_h C_h$.

Differentiating this identity gives

$$p^C_h C_h \left(d \log p^C_h + d \log C_h \right) = \sum_{\substack{h' \in \mathcal{H} \\ i \in \mathcal{N}}} \left(p^Y_{h'i}\, dq_{h,h'i} + q_{h,h'i}\, dp^Y_{h'i} \right).$$

The envelope condition for the consumption aggregator implies

$$p_h^C C_h \, d\log p_h^C = \sum_{\substack{h' \in \mathcal{H} \\ i \in \mathcal{N}}} Y_{h,h'i}^C \, dp_{h'i}^Y.$$

Subtracting this expression and using the definition of net exports yields

$$p_h^C C_h \, d\log C_h = \sum_{\substack{h' \in \mathcal{H} \\ i \in \mathcal{N}}} p_{h'i}^Y \, dq_{h,h'i} + \sum_{\substack{h' \in \mathcal{H} \\ i \in \mathcal{N}}} NX_{h,h'i} \, d\log p_{h'i}^Y. \tag{108}$$

It remains to characterize the first term. At the initial undistorted equilibrium, the production envelope conditions imply

$$\begin{aligned} p_{hi}^Y \, dY_{hi}^Y &= \sum_{\substack{h' \in \mathcal{H} \\ j \in \mathcal{N}}} p_{h'j}^Y \, dY_{hi,h'j}^Y + \sum_{k \in \mathcal{K}} R_{hk} \, dK_{hi,hk} + \sum_{f \in \mathcal{F}} w_{hf} \, dL_{hi,hf}, \\ p_{hk}^X \, dX_{hk} &= \sum_{\substack{h' \in \mathcal{H} \\ i \in \mathcal{N}}} p_{h'i}^Y \, dY_{hk,h'i}^X. \end{aligned}$$

Substituting these conditions into the definition of $q_{h,h'i}$ and summing over goods gives

$$\begin{aligned} \sum_{\substack{h' \in \mathcal{H} \\ i \in \mathcal{N}}} p_{h'i}^Y \, dq_{h,h'i} &= \sum_{\substack{j \in \mathcal{N} \\ k \in \mathcal{K}}} R_{hk} \, dK_{hj,hk} + \sum_{\substack{j \in \mathcal{N} \\ f \in \mathcal{F}}} w_{hf} \, dL_{hj,hf} - \sum_{k \in \mathcal{K}} p_{hk}^X \, dX_{hk} \\ &= \sum_{k \in \mathcal{K}} \left(R_{hk} \, dK_{hk} - p_{hk}^X \, dX_{hk} \right), \end{aligned}$$

where the second equality uses factor-market clearing and $dL_{hf} = 0$. Along the balanced growth path, (6) implies $d\log X_{hk} = d\log K_{hk}$. Therefore,

$$\sum_{\substack{h' \in \mathcal{H} \\ i \in \mathcal{N}}} p_{h'i}^Y \, dq_{h,h'i} = \sum_{k \in \mathcal{K}} \left(R_{hk} K_{hk} - p_{hk}^X X_{hk} \right) d\log K_{hk}.$$

Substituting this expression into (108) and dividing by $p_h^C C_h$ gives

$$d\log C_h = \sum_{k \in \mathcal{K}} \frac{R_{hk} K_{hk} - p_{hk}^X X_{hk}}{p_h^C C_h} d\log K_{hk} + \sum_{\substack{h' \in \mathcal{H} \\ i \in \mathcal{N}}} \frac{NX_{h,h'i}}{p_h^C C_h} d\log p_{h'i}^Y,$$

which is (34).