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A Proofs

Proof of Proposition 1. Since $\partial V/\partial w > 0$ as long as $\boldsymbol{p}(\tau_0) \neq \mathbf{0}$ and $\boldsymbol{R}(\tau_0) > 0$, u is monotone increasing in V. ■

Proofs of Proposition 2 & Proposition 3. These are corollaries of Proposition 4. ■

Proof of Proposition 4. This proof is lengthy, so we break it down into several steps.

1. We define the expenditure function, and discuss the implications of time separability.
2. We prove a series of lemmas about the expenditure function. For rentiers, the expenditure function can be used to calculate the money metric utility function.
3. Finally, we combine the lemmas to prove Proposition 4.

A key challenge, which lengthens the proof, is that because financial markets may be incomplete, we cannot rely on the existence of a single intertemporal budget constraint to solve for money-metric utility.

Throughout this proof, we use a familiar transformation of the utility function. Define the *shadow* intertemporal expenditure function to be

$$e(\boldsymbol{q}, \boldsymbol{\pi}, U) = \min_{\boldsymbol{c}} \{\boldsymbol{q} \cdot \boldsymbol{c} : \boldsymbol{\mathcal{U}}(\boldsymbol{c}, \boldsymbol{\pi}) = U\}, \tag{19}$$

where $\boldsymbol{q}$ has the same dimensionality as the state-contingent commodity space. We refer to e as the *shadow* intertemporal expenditure function and to $\boldsymbol{q}$ as *shadow* prices. We use the qualifier "shadow" to emphasize that e is a purely theoretical construct and agents need not be solving the expenditure minimization problem defined in (19) in practice.

Note that Definition 3 is equivalent to the following.

Proposition 6 (Time-Separability of Expenditure Function)**.** *Suppose that the preference relation $\succeq$ is time-separable in the sense of Definition 3. Then the shadow intertemporal expenditure function must also be* time separable *in the sense that it can be written as*

$$e(\boldsymbol{q}, \boldsymbol{\pi}, U) = \tilde{e}\left(\tilde{P}\left(\boldsymbol{q}(s^0), U\right), \tilde{F}\left(\left\{\boldsymbol{q}(s^t)\right\}_{t>0}, \boldsymbol{\pi}, U\right), U\right), \tag{20}$$

where $\tilde{e}$, $\tilde{P}$, and $\tilde{F}$ are scalar-valued functions. The function $\tilde{e}$ is increasing in all three arguments and homogeneous of degree one in the first two arguments. The functions $\tilde{P}$ and $\tilde{F}$ are increasing and homogeneous of degree one in $\boldsymbol{q}$, and non-decreasing in U.

This proposition is a consequence of Theorem 4.3 in Blackorby et al. (1998).

To prove Proposition 4, we use the shadow intertemporal expenditure function, and proceed by first proving a series of lemmas.

Since each decision problem is indexed by $(\tau, w, \boldsymbol{y})$, denote consumption of good n in history s^t by $c_n(s^t|\tau, w, \boldsymbol{y})$. The next lemma shows that for every decision problem $(\tau, w, \boldsymbol{y})$, there exists a set of shadow prices $\boldsymbol{q}^*(\tau, w, \boldsymbol{y})$ that rationalize the allocations $\boldsymbol{c}(\cdot|\tau, w, \boldsymbol{y})$.

Lemma 2 (Dual Problem)**.** *There exist $\boldsymbol{q}^*(\tau, w, \boldsymbol{y})$ such that, for every s^t and n,*

$$c_n^*(s^t|\boldsymbol{q}^*, \boldsymbol{\pi}, V(\tau, w, \boldsymbol{y})) = c_n(s^t|\tau, w, \boldsymbol{y}),$$

where $c_n^*(s^t|\boldsymbol{q}^*, \boldsymbol{\pi}, V(\tau, w, \boldsymbol{y}))$ are quantities that minimize the shadow expenditure function (19). *Moreover, we can set shadow prices for goods in the first period equal to their observed prices:*

$$q_n^*(s^0|\tau, w, \boldsymbol{y}) = p_n(s^0|\tau),$$

for every $n \in N$.

Proof of Lemma 2. The existence of q^* follows from the separating hyperplane theorem, since the constraint set and indifference curves are both convex (the constraint set is an intersection of convex sets). We can provide a more constructive proof by using the Karush-Kuhn-Tucker (KKT) conditions. Since the expenditure minimization problem is convex, the KKT conditions must be satisfied. The Lagrangian for households is:

$$\begin{aligned}
\mathcal{L}(\boldsymbol{p}, \boldsymbol{R}, \boldsymbol{y}, \boldsymbol{\pi}, w) &= \boldsymbol{\mathcal{U}}(\boldsymbol{c}, \boldsymbol{\pi}) - \lambda(s^0|\tau)[\sum_{n \in N} p_n(s^0|\tau)c_n(s^0|\tau) + \sum_{k \in K} a_k(s^0|\tau) - w] \\
&+ \sum_{s^t} \lambda(s^t|\tau)\left[\sum_{n \in N} p_n(s^t|\tau)c_n(s^t|\tau) + \sum_{k \in K} a_k(s^t|\tau) - \sum_{k \in K} R_k(s^t|\tau)a_k(s^{t-1}|\tau) + y(s^t|\tau)\right] \\
&- \sum_{s^t} \mu(s^t|\tau)\left[\sum_{k} a_k(s^t|\tau) - X(s^t|\tau)\right] \\
&= \boldsymbol{\mathcal{U}}(\boldsymbol{c}, \boldsymbol{\pi}) + \lambda(s^0|\tau)w + \sum_{s^t} \lambda(s^t|\tau)y(s^t|\tau) \\
&- \sum_{s^t = s^0}^{s^T} \lambda(s^t|\tau) \sum_{n \in N} p_n(s^t|\tau)c_n(s^0|\tau)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{s^t=s^0}^{s^T} \lambda(s^t|\tau) \sum_{k\in K} a_k(s^t|\tau) + \sum_{s^t} \lambda(s^t|\tau) \sum_{k\in K} R_k(s^t|\tau) a_k(s^{t-1}|\tau) \\
&- \sum_{s^t} \mu(s^t|\tau) \sum_{k} a_k(s^t|\tau) + \sum_{s^t} \mu(s^t|\tau) X(s^t|\tau)
\end{aligned}
$$

The first order conditions for asset holdings are

$$
-\left[\lambda(s^t|\tau) + \mu(s^t|\tau)\right] = \sum_{s^{t+1}} \lambda(s^{t+1}|\tau) R_k(s^{t+1}|\tau)
$$

Substituting this back in, we get that the Lagrangian is equal to

$$
\mathcal{L}(\boldsymbol{p}, \boldsymbol{R}, \boldsymbol{y}, \boldsymbol{\pi}, w) = \boldsymbol{\mathcal{U}}(\boldsymbol{c}, \boldsymbol{\pi}) + \lambda(s^0|\tau) w + \sum_{s^t} \lambda(s^t|\tau) y(s^t|\tau) - \sum_{s^t=s^0}^{s^T} \lambda(s^t|\tau) \sum_{n\in N} p_n(s^t|\tau) c_n(s^0|\tau) + \sum_{s^t} \mu(s^t|\tau) X(s^t|\tau).
$$

Define the indirect utility function to be v that satisfies this equation:

$$
e(\boldsymbol{q}, \boldsymbol{\pi}, v) = W.
$$

From standard duality, we know that we can also write

$$
v(\boldsymbol{q}, \boldsymbol{\pi}, W) = \max_{\boldsymbol{c}} \{\boldsymbol{\mathcal{U}}(\boldsymbol{c}, \boldsymbol{\pi}) : \boldsymbol{q} \cdot \boldsymbol{c} = W\}.
$$

Call the maximizers above $\boldsymbol{c}^{**}(\boldsymbol{q}, \boldsymbol{\pi}, W)$. The Lagrangian for intertemporal indirect utility function is

$$
\mathcal{L}^{**}(\boldsymbol{q}, \boldsymbol{\pi}, W) = \boldsymbol{\mathcal{U}}\left(\{\boldsymbol{c}, \boldsymbol{\pi}\}\right) - \mu \left[\boldsymbol{q} \cdot \boldsymbol{c} - W\right].
$$

Set

$$
q_n(s^t) = \frac{\lambda(s^t|\tau)}{\lambda(s^0|\tau)} p_n(s^t|\tau)
$$

and

$$
W = w + \sum_{s^t} \frac{\lambda(s^t|\tau)}{\lambda(s^0|\tau)} y(s^t|\tau) + \sum_{s^t} \frac{\mu(s^t|\tau)}{\lambda(s^0|\tau)} X(s^t|\tau).
$$

Hence

$$
\mathcal{L}^{**}(\boldsymbol{q}, \boldsymbol{\pi}, W) = \boldsymbol{\mathcal{U}}\left(\{\boldsymbol{c}, \boldsymbol{\pi}\}\right) + \mu \left[w + \sum_{s^t} \frac{\lambda(s^t|\tau)}{\lambda(s^0|\tau)} y(s^t|\tau) + \sum_{s^t} \frac{\mu(s^t|\tau)}{\lambda(s^0|\tau)} X(s^t|\tau) - \sum_{s^t} \sum_{n\in N} \frac{\lambda(s^t|\tau)}{\lambda(s^0|\tau)} p_n(s^t|\tau) c_n(s^t|\tau) \right].
$$

These problems have the same solution because the Lagrangian is the same. Hence

$$\boldsymbol{c}^{**}(\boldsymbol{q}, \boldsymbol{\pi}, W) = \boldsymbol{c}(s^t|\tau, w, \boldsymbol{y}),$$

where $q_n(s^t) = \lambda(s^t|\tau)p_n(s^t|\tau)$ and $W = \lambda(s^0|\tau)w + \sum_{s^t} \lambda(s^t|\tau)y(s^t|\tau) + \sum_{s^t} \mu(s^t|\tau)X(s^t|\tau)$. By standard duality arguments, we also know that

$$\boldsymbol{c}^{**}(\boldsymbol{q}, \boldsymbol{\pi}, W) = \boldsymbol{c}^{*}(\boldsymbol{q}, \boldsymbol{\pi}, v(\boldsymbol{q}, \boldsymbol{\pi}, W)) = \boldsymbol{c}^{*}(\boldsymbol{q}, \boldsymbol{\pi}, V(\boldsymbol{q}, \boldsymbol{\pi}, W)).$$

■

Next, define the following function, called the compensated budget share of n:

$$b_n(\boldsymbol{q}(s^0), U) = \frac{c_n(s^0)q_n(s^0)}{e(\boldsymbol{q}, \boldsymbol{\pi}, U)b^P(\boldsymbol{q}, \boldsymbol{\pi}, U)},$$

where we suppress the dependence of $c_n(s^0)$ on $(\boldsymbol{q}(s^0), U)$.

Lemma 3. *If preferences are time separable, then the following holds*

$$b^P(\boldsymbol{q}, \boldsymbol{\pi}, U) \equiv \sum_{n \in N} \frac{c_n(s^0)q_n(s^0)}{e(\boldsymbol{q}, \boldsymbol{\pi}, U)} = \frac{\partial \log e(\boldsymbol{q}, \boldsymbol{\pi}, U)}{\partial \log \tilde{P}},$$

$$b^F(\boldsymbol{q}, \boldsymbol{\pi}, U) \equiv 1 - b^P(\boldsymbol{q}, \boldsymbol{\pi}, U) = \frac{\partial \log e(\boldsymbol{q}, \boldsymbol{\pi}, U)}{\partial \log \tilde{F}},$$

and

$$\frac{\partial \log \tilde{P}}{\partial \log q_n(s^0)} = \frac{c_n(s^0)q_n(s^0)}{e(\boldsymbol{q}, \boldsymbol{\pi}, U)b^P(\boldsymbol{q}, \boldsymbol{\pi}, U)}.$$

Proof. By the envelope theorem,

$$\frac{\partial \log e(\boldsymbol{q}, \boldsymbol{\pi}, U)}{\partial \log q_n(s^0)} = \frac{c_n(s^0)q_n(s^0)}{e(\boldsymbol{q}, \boldsymbol{\pi}, U)},$$

and for $t > 0$

$$\frac{\partial \log e(\boldsymbol{q}, \boldsymbol{\pi}, U)}{\partial \log q_n(s^t)} = \frac{c_n(s^t)q_n(s^t)}{e(\boldsymbol{q}, \boldsymbol{\pi}, U)}.$$

Then, we know that

$$b^P(\boldsymbol{q}, \boldsymbol{\pi}, U) = \sum_{n \in N} \frac{c_n(s^0)q_n(s^0)}{e(\boldsymbol{q}, \boldsymbol{\pi}, U)} = \sum_{n \in N} \frac{\partial \log e(\boldsymbol{q}, \boldsymbol{\pi}, U)}{\partial \log q_n(s^0)} = \frac{\partial \log e(\boldsymbol{q}, \boldsymbol{\pi}, U)}{\partial \log \tilde{P}} \sum_{n \in N} \frac{\partial \log \tilde{P}}{\partial \log q_n(s^0)} = \frac{\partial \log e(\boldsymbol{q}, \boldsymbol{\pi}, U)}{\partial \log \tilde{P}}$$

and

$$b^F(\boldsymbol{q},\boldsymbol{\pi},U) = \sum_{s^t|t>0}\sum_{n\in N}\frac{c_n(s^t)q_n(s^t)}{e(\boldsymbol{q},\boldsymbol{\pi},U)} = \sum_{s^t|t>0}\sum_{n\in N}\frac{\partial\log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial\log q_n(s^t)}$$
$$= \frac{\partial\log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial\log\tilde{F}}\sum_{s^t|t>0}\sum_{n\in N}\frac{\partial\log\tilde{F}}{\partial\log q_n(s^0)} = \frac{\partial\log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial\log\tilde{F}}$$

where the last steps use homogeneity of degree 1 in $\boldsymbol{q}$ of $\tilde{P}$ and $\tilde{F}$.

Next, we show that

$$\frac{\partial\log\tilde{P}}{\partial\log q_n(s^0)} = b_n(\boldsymbol{q}(s^0),U).$$

To do this, use the following equality,

$$\begin{aligned}\frac{\partial\log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial\log q_n(s^0)} &= \frac{\partial\log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial\log\tilde{P}}\frac{\partial\log\tilde{P}}{\partial\log q_n(s^0)}\\ &= \frac{c_n(s^0)q_n(s^0)}{e(\boldsymbol{q},\boldsymbol{\pi},U)}\\ &= b^P(\boldsymbol{q},\boldsymbol{\pi},U)\frac{\partial\log\tilde{P}}{\partial\log q_n(s^0)}.\end{aligned}$$

Rearranging yields

$$\frac{\partial\log\tilde{P}}{\partial\log q_n(s^0)} = \frac{c_n(s^0)q_n(s^0)}{e(\boldsymbol{q},\boldsymbol{\pi},U)b^P(\boldsymbol{q},\boldsymbol{\pi},U)}.$$

■

Lemma 4. *When preferences are time separable, the elasticity of intertemporal substitution*

$$\sigma^*(\boldsymbol{q},\boldsymbol{\pi},U) \equiv 1 - \sum_{n\in N}\frac{\partial\log b^p(q,\pi,U)/b^F(q,\pi,U)}{\partial\log q_n(s^0)}.$$

is given by

$$\sigma^*(\boldsymbol{q},\boldsymbol{\pi},U) = 1 - \frac{\partial^2\log e/\left(\partial\log\tilde{P}\right)^2}{b^F(\boldsymbol{q},\boldsymbol{\pi},U)b^P(\boldsymbol{q},\boldsymbol{\pi},U)}.$$

Proof. We start with

$$\frac{\partial\log b^P(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial\log q_n(s^0)} = \frac{1}{b^P(\boldsymbol{q},\boldsymbol{\pi},U)}\frac{\partial}{\partial\log q_n(s^0)}\left[\sum_{k\in N}\frac{\partial\log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial\log\tilde{P}}\frac{\partial\log\tilde{P}}{\partial\log q_k(s^0)}\right],$$

$$
\begin{aligned}
&= \frac{1}{b^P(\boldsymbol{q},\boldsymbol{\pi},U)} \frac{\partial}{\partial \log q_n(s^0)} \left[\sum_{k\in N} \frac{\partial \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \log \tilde{P}} b_k(\boldsymbol{q},U) \right], \\
&= \frac{1}{b^P(\boldsymbol{q},\boldsymbol{\pi},U)} \left[\sum_{k\in N} \frac{\partial}{\partial \log q_n(s^0)} \frac{\partial \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \log \tilde{P}} b_k(\boldsymbol{q},U) + \sum_{k\in N} \frac{\partial \log e}{\partial \log \tilde{P}} \frac{\partial b_k(\boldsymbol{q},U)}{\partial \log q_n(s^0)} \right], \\
&= \frac{1}{b^P(\boldsymbol{q},\boldsymbol{\pi},U)} \left[\sum_{k\in N} \frac{\partial^2 \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{(\partial \log \tilde{P})^2} b_n(\boldsymbol{q},U) b_k(\boldsymbol{q},U) + \frac{\partial \log e}{\partial \log \tilde{P}} \frac{\sum_{k\in N} \partial b_k(\boldsymbol{q},U)}{\partial \log q_n(s^0)} \right], \\
&= \frac{1}{b^P(\boldsymbol{q},\boldsymbol{\pi},U)} \left[\frac{\partial^2 \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{(\partial \log \tilde{P})^2} b_n(\boldsymbol{q},U) \sum_{k\in N} b_k(\boldsymbol{q},U) \right], \\
&= \frac{1}{b^P(\boldsymbol{q},\boldsymbol{\pi},U)} \frac{\partial^2 \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{(\partial \log \tilde{P})^2} b_n(\boldsymbol{q},U).
\end{aligned}
$$

Summing over all $n \in N$ yields

$$
\begin{aligned}
\sum_n \frac{\partial \log b^P(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \log q_n(s^0)} &= \frac{1}{b^P(\boldsymbol{q},\boldsymbol{\pi},U)} \frac{\partial^2 \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{(\partial \log \tilde{P})^2} \sum_n b_n(\boldsymbol{q},U), \\
&= \frac{1}{b^P(\boldsymbol{q},\boldsymbol{\pi},U)} \frac{\partial^2 \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{(\partial \log \tilde{P})^2}.
\end{aligned}
$$

Since $b^P + b^F = 1$,we have that

$$
\frac{\partial \log b^F(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \log q_n(s^0)} = -\frac{b^P(\boldsymbol{q},\boldsymbol{\pi},U)}{b^F(\boldsymbol{q},\boldsymbol{\pi},U)} \frac{\partial \log b^P(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \log q_n(s^0)}.
$$

By definition,

$$
\begin{aligned}
1 - \sigma(\boldsymbol{q},\boldsymbol{\pi},U) &= \sum_n \frac{\partial \log b^P(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \log q_n(s^0)} - \sum_n \frac{\partial \log b^F(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \log q_n(s^0)}, \\
&= \frac{1}{b^P(\boldsymbol{q},\boldsymbol{\pi},U)} \frac{\partial^2 \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{(\partial \log \tilde{P})^2} + \frac{b^P(\boldsymbol{q},\boldsymbol{\pi},U)}{b^F(\boldsymbol{q},\boldsymbol{\pi},U)} \frac{\partial \log b^P(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \log q_n(s^0)}, \\
&= \frac{1}{b^F(\boldsymbol{q},\boldsymbol{\pi},U) b^P(\boldsymbol{q},\boldsymbol{\pi},U)} \frac{\partial^2 \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{(\partial \log \tilde{P})^2}.
\end{aligned}
$$

■

Lemma 5. *When preferences are time separable, the following equation holds:*

$$
\frac{\partial \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \log \boldsymbol{q}} d\log \boldsymbol{q} + \frac{\partial \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \boldsymbol{\pi}} d\boldsymbol{\pi} = -\frac{d \log b^P(\boldsymbol{q},\boldsymbol{\pi},U)}{1 - \sigma^*(\boldsymbol{q},\boldsymbol{\pi},U)} + \sum_{n\in N} b_n(\boldsymbol{q}(s^0),U) d\log q_n(s^0)
$$

Proof. From Lemma 3, we know that

$$\frac{\partial \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \log \boldsymbol{q}} d\log \boldsymbol{q} + \frac{\partial \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \boldsymbol{\pi}} d\boldsymbol{\pi} = b^P(\boldsymbol{q},\boldsymbol{\pi},U)\sum_{n\in N} b_n(\boldsymbol{q},U) d\log \boldsymbol{q}(s^0)$$
$$+b^F(\boldsymbol{q},\boldsymbol{\pi},U)\sum_{s^t|t>0}\left(\sum_{n\in N}\frac{\partial \log F}{\partial \log q_n(s^t)} d\log q_n(s^t) + \frac{\partial \log F}{\partial \boldsymbol{\pi}(s^t)} d\boldsymbol{\pi}(s^t)\right).$$

Next, from homogeneity of degree one, we know that

$$\frac{\partial \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \log \tilde{P}} + \frac{\partial \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \log \tilde{F}} = 1.$$

Differentiating this identity with respect to P and F yields the following equation

$$\frac{\partial^2 \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\left(\partial \log \tilde{P}\right)^2} = -\frac{\partial^2 \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\partial \log \tilde{P} \partial \log \tilde{F}} = \frac{\partial^2 \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{\left(\partial \log \tilde{F}\right)^2}.$$

Hence, fixing utility, the total derivative of $b^P(\boldsymbol{q},\boldsymbol{\pi},U)$ with respect to $\boldsymbol{q}$ and $\boldsymbol{\pi}$ is

$$\begin{aligned} b^P d\log b^P(\boldsymbol{q},\boldsymbol{\pi},U) &= \frac{\partial^2 \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{(\partial \log \tilde{P})^2}\sum_{n\in N}\frac{\partial \log \tilde{P}}{\partial \log q_n(s^0)} d\log q_n(s^0) \\ &+ \frac{\partial^2 \log e}{\partial \log \tilde{F} \partial \log \tilde{P}}\sum_{s^t|t>0}\left(\sum_{n\in N}\frac{\partial \log \tilde{F}}{\partial \log q_n(s^t)} d\log q_n(s^t) + \frac{\partial \log \tilde{F}}{\partial \pi(s^t)} d\pi(s^t)\right) \\ &= \frac{\partial^2 \log e(\boldsymbol{q},\boldsymbol{\pi},U)}{(\partial \log \tilde{P})^2}\left[\begin{array}{c} \sum_{n\in N} b_n(\boldsymbol{q},U) d\log q_n(s^0) - \\ \sum_{s^t|t>0}\left(\sum_{n\in N}\frac{\partial \log \tilde{F}}{\partial \log q_n(s^t)} d\log q_n(s^t) + \frac{\partial \log \tilde{F}}{\partial \pi(s^t)} d\pi(s^t)\right)\end{array}\right]. \end{aligned} \tag{21}$$

From Lemma 3 and Lemma 4, we can rewrite this as

$$\frac{d\log b^P(\boldsymbol{q},\boldsymbol{\pi},U)}{(1-\sigma^*(\boldsymbol{q},\boldsymbol{\pi},U))} = (1-b^P(\boldsymbol{q},\boldsymbol{\pi},U))\left[\begin{array}{c} \sum_{n\in N} b_n(\boldsymbol{p},U) d\log q_n(s^0) - \\ \sum_{s^t|t>0}\left(\sum_{n\in N}\frac{\partial \log \tilde{F}}{\partial \log q_n(s^t)} d\log q_n(s^t) + \frac{\partial \log \tilde{F}}{\partial \pi(s^t)} d\pi(s^t)\right)\end{array}\right],$$

Rearranging this gives

$$\begin{array}{c} b^P(\boldsymbol{q},\boldsymbol{\pi},U)\sum_{n\in N} b_n(\boldsymbol{p},U) d\log q_n(s^0) - b^F(\boldsymbol{q},\boldsymbol{\pi},U))\times \\ \sum_{s^t|t>0}\left(\sum_{n\in N}\frac{\partial \log \tilde{F}}{\partial \log q_n(s^t)} d\log q_n(s^t) + \frac{\partial \log \tilde{F}}{\partial \pi(s^t)} d\pi(s^t)\right)\end{array} = -\frac{d\log b^P(\boldsymbol{q},\boldsymbol{\pi},U)}{1-\sigma^*(\boldsymbol{q},\boldsymbol{\pi},U)} + \sum_{n\in N} b_n(\boldsymbol{q}(s^0),U) d\log q_n(s^0).$$

Plug this back into (21) to get the desired result. ■

Lemma 6. *The shadow prices $\boldsymbol{q}^*(\tau, w, \mathbf{0})$ can be written as a function of τ and $V(\tau, w, \mathbf{0})$. That is, we can write*

$$\boldsymbol{q}^*(\tau, w, \mathbf{0}) = \boldsymbol{q}^*\left(\tau, V(\tau, w, \mathbf{0})\right).$$

Furthermore,

$$V(\tau, w, \mathbf{0}) = v(\boldsymbol{q}^*\left(\tau, V(\tau, w, \mathbf{0})\right), \boldsymbol{\pi}(\cdot|\tau), w)$$

for every τ and w.

Proof. The first part follows from the fact that the value function $V(\tau, w, \mathbf{0})$ is monotone in w. Hence, we can substitute the inverse of $V(\tau, w, \mathbf{0})$ with respect to w into $\boldsymbol{q}^*(\tau, w, \mathbf{0})$ to get $\boldsymbol{q}^*\left(\tau, V(\tau, w, \mathbf{0})\right) = \boldsymbol{q}^*(\tau, V^{-1}(V(\tau, w, \mathbf{0})), \mathbf{0})$.

For the second part, we know from Proposition 2, that

$$\boldsymbol{c}(\tau, w, \mathbf{0}) = \boldsymbol{c}^*(\boldsymbol{q}^*(\tau, V(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau), V(\tau, w, \mathbf{0})).$$

Hence

$$\begin{aligned} V(\tau, w, \mathbf{0}) &= \boldsymbol{\mathcal{U}}(\boldsymbol{c}(\tau, w, \mathbf{0}), \boldsymbol{\pi}(\cdot|\tau)) \\ &= \boldsymbol{\mathcal{U}}(\boldsymbol{c}^*(\boldsymbol{q}^*(\tau, V(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau), V(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau) \\ &= v(\boldsymbol{q}^*\left(\tau, V(\tau, w, \mathbf{0})\right), \boldsymbol{\pi}(\cdot|\tau), w). \end{aligned}$$

■

We now narrow our focus on rentiers. The following lemma shows that for the rentiers, the expenditure function, evaluated at shadow prices and the intertemporal indifference curve $m(\tau, w, \mathbf{0})$ coincides with that level of financial wealth.

Lemma 7. *The following holds*

$$e(\boldsymbol{q}^*\left(\tau, m(\tau, w, \mathbf{0})\right), \boldsymbol{\pi}(\cdot|\tau), m(\tau, w, \mathbf{0})) = w.$$

Proof. From the proof of Proposition 2, we know that

$$e(\boldsymbol{q}^*\left(\tau, m(\tau, w, \boldsymbol{y})\right), \boldsymbol{\pi}(\cdot|\tau), m(\tau, w, \boldsymbol{y})) = w + \sum_{s^t} \lambda(s^t|\tau) y(s^t|\tau) + \sum_{s^t} \mu(s^t|\tau) X(s^t|\tau),$$

where $\lambda(s^t|\tau)$ are lagrange multipliers on state-contingent budget constraints and $\mu(s^t|\tau)$

are lagrange multipliers on borrowing constraints. Since $y(s^t|\tau) = 0$, we know that

$$e(\boldsymbol{q}^*(\tau, m(\tau, w, \boldsymbol{y})), \boldsymbol{\pi}(\cdot|\tau), m(\tau, w, \boldsymbol{y})) = w + \sum_{s^t} \mu(s^t|\tau) X(s^t|\tau).$$

We prove the desired result by showing that $\mu(s^t) \equiv 0$. To do this, we use backward induction. Suppose that for some t, we know that, for every $t' > t$, we have $\sum_k a_k(s^{t'}|\tau) \geq 0$. That is, the borrowing constraint is slack for every $s^{t'}$ following s^t. For the sake of deriving a contradiction, suppose that $\mu(s^t|\tau) \neq 0$. Then

$$\sum_{n \in N} p_n(s^{t+1}|\tau) c_n(s^{t+1}|\tau) + \sum_k a_k(s^{t+1}|\tau) = \sum_{k \in K} R_k(s^t|\tau) a_k(s^{t-1}|\tau) < -\left[\min_k R_k(s^T|\tau)\right] X(s^{T-1}|\tau) < 0.$$

This implies that

$$\sum_k a_k(s^{t+1}|\tau) < 0,$$

which is a contradiction. Hence, we know that

$$\sum_k a_k(s^{t+1}|\tau) \geq 0.$$

This implies that $\mu(s^t|\tau) = 0$. We finish by observing that we know that for every s^T, the no-Ponzi scheme condition implies that

$$\sum_k a_k(s^T|\tau) \geq 0.$$

This is the first step of the backward induction.

■

With these preliminaries out of the way, we are ready to prove Proposition 4. We start with the definition of the money-metric. That is, $m(\tau, w, \mathbf{0})$ solves the following equation:

$$V(\tau, w, \mathbf{0}) = V(\tau_0, m(\tau, w, \mathbf{0}), \mathbf{0}).$$

From Lemma 6, we know

$$v(\boldsymbol{q}^*(\tau, V(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau), w) = V(\tau, w, \mathbf{0}) = V(\tau_0, m(\tau, w, \mathbf{0}), \mathbf{0}) = v(\boldsymbol{q}^*(\tau_0, V(\tau, w, \mathbf{0}), \mathbf{0}), \boldsymbol{\pi}(\cdot|\tau_0), m(\tau, w, \mathbf{0})).$$

Hence, $m(\tau, w, \mathbf{0})$ solves

$$v(\boldsymbol{q}^*(\tau, V(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau), w) = v(\boldsymbol{q}^*(\tau_0, V(\tau, w, \mathbf{0}), \mathbf{0}), \boldsymbol{\pi}(\cdot|\tau_0), m(\tau, w, \mathbf{0})).$$

Without loss of generality, by Proposition 1, cardinalize the value function using the money-metric (since the value function is only defined up to monotone transformations). Therefore

$$v(\boldsymbol{q}^*(\tau, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau), w) = v(\boldsymbol{q}^*(\tau_0, m(\tau, w, \mathbf{0}), \mathbf{0}), \boldsymbol{\pi}(\cdot|\tau_0), m(\tau, w, \mathbf{0})).$$

Using the shadow expenditure function, we can write

$$\begin{aligned}
m(\tau, w, \mathbf{0}) =& e(\boldsymbol{q}^*(\tau_0, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau_0), m(\tau, w, \mathbf{0})),\\
=& e(\boldsymbol{q}^*(\tau_0, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau_0), m(\tau, w, \mathbf{0}))\frac{e(\boldsymbol{q}^*(\tau, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau), m(\tau, w, \mathbf{0}))}{e(\boldsymbol{q}^*(\tau, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau), m(\tau, w, \mathbf{0}))},\\
=& e(\boldsymbol{q}^*(\tau, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau), m(\tau, w, \mathbf{0}))\frac{e(\boldsymbol{q}^*(\tau_0, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau_0), m(\tau, w, \mathbf{0}))}{e(\boldsymbol{q}^*(\tau, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau), m(\tau, w, \mathbf{0}))},\\
=& w\frac{e(\boldsymbol{q}^*(\tau_0, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau_0), m(\tau, w, \mathbf{0}))}{e(\boldsymbol{q}^*(\tau, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau), m(\tau, w, \mathbf{0}))},
\end{aligned}$$

where the last line uses Lemma 7. Logging both sides gives

$$\begin{aligned}
\log m(\tau, w, \mathbf{0}) =& \log w + \log\frac{e(\boldsymbol{q}^*(\tau_0, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau_0), m(\tau, w, \mathbf{0}))}{e(\boldsymbol{q}^*(\tau, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|\tau), m(\tau, w, \mathbf{0}))},\\
=& \log w + \int_{\tau}^{\tau_0}\Bigg(\frac{\partial \log e(\boldsymbol{q}^*(x, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|x), m(\tau, w, \mathbf{0}))}{\partial \log \boldsymbol{q}^*}\frac{d\log \boldsymbol{q}^*}{dx}\\
&+ \frac{\partial \log e(\boldsymbol{q}^*(x, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|x), m(\tau, w, \mathbf{0}))}{\partial \log \boldsymbol{\pi}(\cdot|x)}\frac{d\log \boldsymbol{\pi}(\cdot|x)}{dx}\Bigg)dx,
\end{aligned}$$

where the second equality uses the fundamental theorem of calculus for line integrals. Using Lemma 5, we can rewrite the last line as

$$\begin{aligned}
\log m(\tau, w, \mathbf{0}) =& \log w - \int_{\tau_0}^{\tau}\Bigg(\sum_{n\in N} b_n(\boldsymbol{p}(\cdot|x), m(\tau, w, \mathbf{0}))\frac{d\log p_n}{dx}\\
&+\frac{d\log b^P(\boldsymbol{q}^*(x, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|x), m(\tau, w, \mathbf{0}))}{\sigma^*(\boldsymbol{q}^*(\tau, m(\tau, w, \mathbf{0})), \boldsymbol{\pi}(\cdot|x), m(\tau, w, \mathbf{0})) - 1}\frac{1}{dx}\Bigg)dx,\\
=& \log w - \int_{\tau_0}^{\tau}\left(\sum_{n\in N} B_n(x, w_x^*, \mathbf{0})\frac{d\log p_n}{dx} + \frac{1}{\sigma(x, w_x^*, \mathbf{0}) - 1}\frac{d\log B^P(x, w_x^*, \mathbf{0})}{dx}\right)dx,
\end{aligned}$$

where for the last step, we replaced compensated budget share with uncompensated budget share for wealth w^*_x, defined in the statement of the proposition. This completes the proof of Proposition 4.

■

Proof of Lemma 1. We need to show that

$$B_n(\tau, w, \boldsymbol{y}) = b_n(\boldsymbol{p}(s^0|\tau), V(\tau, w, \boldsymbol{y})).$$

By Lemma 2, we know that

$$B_n(\tau, w, \boldsymbol{y}) = \frac{p_n(s^0|\tau)c_n(s^0|\tau, w, \boldsymbol{y})}{\sum_{m\in N} p_m(s^0|\tau)c_m(s^0|\tau, w, \boldsymbol{y})} = \frac{q^*_n(s^0)c^*_n(s^0|\boldsymbol{q}^*, \boldsymbol{\pi}, V(\tau, w, \boldsymbol{y}))}{\sum_{m\in N} q_m(s^0|\tau)c^*_m(s^0|\boldsymbol{q}^*, \boldsymbol{\pi}, V(\tau, w, \boldsymbol{y}))} \equiv b_n(\boldsymbol{q}^*, \boldsymbol{\pi}, V(\tau, w, \boldsymbol{y})).$$

Next, we know, from Shephard's lemma that for each $n \in N$

$$\begin{aligned}\frac{q^*_n(s^0)c^*_n(\boldsymbol{q}^*, \boldsymbol{\pi}, V(\tau, w, \boldsymbol{y}))}{e(\boldsymbol{q}^*, \boldsymbol{\pi}, V(\tau, w, \boldsymbol{y}))} =& \frac{\partial \log e(\boldsymbol{q}^*, \boldsymbol{\pi}, V(\tau, w, \boldsymbol{y}))}{\partial \log q^*_n(s^0)} \\ =& \frac{\partial \log e\left(P\left(\boldsymbol{q}^*(s^0), V(\tau, w, \boldsymbol{y})\right), F\left(\{\boldsymbol{q}^*(s^t)\}_{t>0}, \boldsymbol{\pi}, V(\tau, w, \boldsymbol{y})\right), V(\tau, w, \boldsymbol{y})\right)}{\partial \log P} \\ & \frac{\partial \log P(\boldsymbol{q}^*(s^0), V(\tau, w, \boldsymbol{y}))}{\partial \log q^*_n(s^0)}.\end{aligned}$$

Hence, we have that

$$\frac{q^*_n(s^0)c^*_n(s^0|\boldsymbol{q}^*, \boldsymbol{\pi}, V(\tau, w, \boldsymbol{y}))}{\sum_{m\in N} q_m(s^0|\tau)c^*_m(s^0|\boldsymbol{q}^*, \boldsymbol{\pi}, V(\tau, w, \boldsymbol{y}))} = \frac{\frac{\partial \log P(\boldsymbol{q}^*(s^0), V(\tau, w, \boldsymbol{y}))}{\partial \log q^*_n(s^0)}}{\sum_{m\in N} \frac{\partial \log P(\boldsymbol{q}^*(s^0), V(\tau, w, \boldsymbol{y}))}{\partial \log q^*_m(s^0)}},$$

which is only a function of $\boldsymbol{q}^*(s^0) = \boldsymbol{p}(s^0|\tau)$ and $V(\tau, w, \boldsymbol{y})$ as needed. ■

Proof of Proposition 5. From Lemma 1, we know that

$$\boldsymbol{B}(\tau, w, \boldsymbol{y}) = b_n(\boldsymbol{p}(s^0|\tau), V(\tau, w, \boldsymbol{y})).$$

By definition of $m(\tau, w, \boldsymbol{y}|\tau)$, it follows that

$$\boldsymbol{B}(\tau, w, \boldsymbol{y}) = b_n(\boldsymbol{p}(s^0|\tau), V(\tau, m(\tau, w, \boldsymbol{y}|\tau), \boldsymbol{0})).$$

Since b is an injective function, we can write

$$V(\tau, m(\tau, w, \boldsymbol{y}|\tau), \mathbf{0})) = b_n^{-1}\left(\boldsymbol{p}(s^0|\tau), \boldsymbol{B}(\tau, w, \boldsymbol{y})\right).$$

Since V is monotone in wealth, we can write

$$m(\tau, w, \boldsymbol{y}|\tau) = V^{-1}\left(\tau, b_n^{-1}\left(\boldsymbol{p}(s^0|\tau), \boldsymbol{B}(\tau, w, \boldsymbol{y})\right), \mathbf{0}\right) = m(\boldsymbol{B}(\tau, w, \boldsymbol{y}), \tau).$$

$\blacksquare$

Proof of Corollary 1. Lemma 1 shows that

$$B_i(\tau, w, \boldsymbol{y}) = b_i(\boldsymbol{p}(s^0|\tau), V(\tau, w, \boldsymbol{y})).$$

Hence, if b_i is monotone in V, then

$$B_i(\tau, w, \boldsymbol{y}) = b_i(\boldsymbol{p}(s^0|\tau), V(\tau, w, \boldsymbol{y})) = b_i(\boldsymbol{p}(s^0|\tau), V(\tau, w^*, \mathbf{0})) = B_i(\tau, w^*, \mathbf{0})$$

if, and only if,

$$V(\tau, w, \boldsymbol{y}) = V(\tau, w^*, \mathbf{0}).$$

$\blacksquare$

B Appendix for Section 2: General Environment and Measure of Welfare

B.1 Non-Separable Preferences

Example 1 provides examples of preferences that satisfy the time separability condition in Definition 3. In this appendix, we provide some examples that do not satisfy this condition.

Example 3 (Examples Violating Time-Separability)**.** Dynamic Stone-Geary:

$$U = \frac{(c(s^0) + a_0)^{1-\sigma}}{1-\sigma} + \sum_{j=1}^{J} \beta^j \frac{(c(s^j) + a_j)^{1-\sigma}}{1-\sigma},$$

and dynamic Addilog:

$$U = \frac{c(s^0)^{1-\sigma_0}}{1-\sigma_0} + \sum_{j=1}^{J} \beta^j \frac{c(s^j)^{1-\sigma_j}}{1-\sigma_j},$$

are not time-separable unless $a_j = 0$ and $\sigma_j = \sigma$ for every j.

The direct utility function of these preferences is separable across time, in the sense that utility can be written as a function of a present block and a future block. That is, one can write $U = \tilde{D}\left(\tilde{P}(c(s^0)), \tilde{F}\left(\{c(s^j)\}_{j>0}\right)\right)$. However, these preferences are not time-separable in the sense of Definition 3 since the functions $\tilde{D}$, $\tilde{P}$, and $\tilde{F}$ will not be homogeneous of degree one.

In the language of utility theory from Blackorby et al. (1998), the examples above are *directly separable* but they are not *separable in the expenditure/distance* function. The different notions of separability—separability in the direct utility function, separability in the indirect utility function, and separability in the expenditure/distance function— do not nest each other and have different implications for separability of the Frisch, Marshallian, and Hicksian demand systems, respectively. They do, however, coincide if, and only if, preferences are homothetic.

Although the examples above are not time separable in the sense defined in Definition 3, they can be modified to become time-separable. For example, the following "pseudo" Stone-Geary functional form is time separable:

$$U = \frac{(c(s^0) + a_0)^{1-\sigma}}{1-\sigma} + \frac{\left[\left(\sum_{j=1}^{J} \beta_j c(s^j)^{1-\sigma}\right)^{\frac{1}{1-\sigma}} + a_1\right]^{1-\sigma}}{1-\sigma}.$$

Similarly, the following "pseudo" Addilog functional form is also time-separable:

$$U = \frac{c(s^0)^{1-\sigma_0}}{1-\sigma_0} + \frac{F\left(\left\{c(s^j)\right\}_{j>0}\right)^{1-\sigma_1}}{1-\sigma_1},$$

as long as F is homogeneous of degree one.

Another example that does not satisfy time separability in Definition 3 is the intertemporal non-homothetic CES preferences studied in Comin et al. (2021). We discuss the theoretical implications and quantitative performance of our method under these preferences in Sections 4.3 and 5.

C Appendix for Section 3: Main Results

C.1 Implementation Details for the Rentier Fixed Point

This appendix provides the implementation details behind Proposition 4. The proposition is stated as a fixed point because the compensated objects entering (13) must be evaluated along the same intertemporal indifference curve as the target household.

Solution Method. To apply Proposition 4, we can follow the same procedures explained in detail in Baqaee et al. (2024). We begin by guessing a candidate solution $m^0(\tau, w, \mathbf{0})$, for example, by deflating nominal wealth at each date using the chained inflation index and ignoring the role of the future. We then use this initial guess on the right-hand side of (13) to get a new guess. We then iterate on this until convergence.

Boundaries. Proposition 4 can only be applied inside a suitable boundary. This is because the budget shares $\boldsymbol{B}(\tau, w, \mathbf{0})$ and consumption-to-wealth ratios $B^P(\tau, w, \mathbf{0})$ are observed only for some subset of wealth levels, say $w \in [\underline{w}_x, \overline{w}_x]$ for $x \in [\tau_0, \tau]$, within each cohort. This limits the range of values of w for which we can calculate the money-metric without out-of-sample extrapolation. Intuitively, for a household in a given cohort at date τ and wealth w, if the money-metric value $m(\tau, w, \mathbf{0})$ is not in $[m(x, \underline{w}_x, \mathbf{0}), m(x, \overline{w}_x, \mathbf{0})]$ for some $x \in [\tau_0, \tau]$, then we cannot recover $m(\tau, w, \mathbf{0})$ without extrapolation. This is because there are no households in that cohort at date x that are on the same indifference curve as the rentier with wealth w at time τ.

Fortunately, Proposition 4 automatically provides the boundary over which the money-metric can be calculated without extrapolation. The initial boundary at τ_0 is just the range in the data: $[\underline{w}_{\tau_0}, \overline{w}_{\tau_0}]$. As we solve (13) forward, for each $\tau > \tau_0$, we can update the boundary because the information required to compute it only depends on previous values of the money-metric (see Baqaee et al., 2024 for a discussion of this issue in a static context).

D Appendix for Section 4: Extensions

We first discuss additional extensions to time inconsistency, labor-leisure choice, and changes in mortality risk. We then provide the details of the non time-separable case.

D.1 Other Extensions

This appendix collects extensions discussed briefly in Section 4 but not kept in the main text.

D.1.1 Time-Inconsistent Preferences

In defining the consumers' value function, we assume that the consumer makes decisions as if the consumption plan chosen in the first period is followed in subsequent periods. If preferences are time inconsistent and the household cannot commit, future selves may instead choose different future consumption plans when those dates arrive. This does not by itself invalidate our sufficient-statistic formulas. As long as time separability holds, in the sense that relative spending in the present depends only on current relative prices and U, and relative spending chosen by future selves depends only on future relative prices and the same U, Proposition 4 and Proposition 5 continue to apply.

To see the logic in a simple case, suppose preferences are of the form

$$U^{\frac{\sigma(U)-1}{\sigma(U)}} = \left(\sum_{n'} \omega_{n'0} U^{\epsilon_{n'0}} c_{n'0}^{\frac{\gamma-1}{\gamma}}\right)^{\frac{\gamma}{\gamma-1}\frac{\sigma(U)-1}{\sigma(U)}} + \beta(U) F\left(\boldsymbol{c}_{j>0}, U\right)^{\frac{\sigma(U)-1}{\sigma(U)}},$$

where F is homogeneous of degree one. Assume further that there is no risk and there is a single risk-free interest rate between each period. The current self expects future selves to choose consumption according to

$$\boldsymbol{c}_{nj>0} = f_{nj}\left(\boldsymbol{p}_{j>0}, U\right) \frac{w_1}{p_{nj}},$$

where w_1 is wealth at the first future period and f_{nj} is the budget share on good n in future period $j > 0$, with

$$\sum_{n,j>0} f_{nj}\left(\boldsymbol{p}_{j>0}, U\right) = 1.$$

These future choices need not coincide with the choices the household would have made under commitment. The key restriction is that the future choices expected by the current self depend only on future relative prices and U.

The household in the present has initial wealth w and chooses savings a_1 subject to

$$\sum_{n'} p_{n'0} c_{n'0} + a_1 = w,$$

with next-period wealth $w_1 = r_1 a_1$. Replacing future choices into the utility function, after optimizing the within-period allocation of current consumption, yields

$$U^{\frac{\sigma(U)-1}{\sigma(U)}}=\left(\frac{w-a_1}{\left(\sum_{n'}\omega_{n'0}^{\gamma}U^{\epsilon_{n'}\gamma}p_{n'0}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}}\right)^{\frac{\sigma(U)-1}{\sigma(U)}}+\beta(U)\left(r_1a_1G\left(\boldsymbol{p}_{j>0},U\right)\right)^{\frac{\sigma(U)-1}{\sigma(U)}}, \tag{22}$$

where

$$r_1a_1G\left(\boldsymbol{p}_{j>0},U\right)=F\left(f_{nj}\left(\boldsymbol{p}_{j>0},U\right)\frac{w_1}{p_{ns}},U\right)=r_1a_1F\left(f_{nj}\left(\boldsymbol{p}_{j>0},U\right)\frac{1}{p_{ns}},U\right).$$

The current self chooses a_1 to maximize U. The first-order condition is

$$\left(\frac{w-a_1}{\left(\sum_{n'}\omega_{n'0}^{\gamma}U^{\epsilon_{n'}\gamma}p_{n'0}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}}\right)^{\frac{\sigma(U)-1}{\sigma(U)}}(w-a_1)^{-\frac{1}{\sigma}}=\beta(U)\,r_1^{1-\frac{1}{\sigma}}a_1^{-\frac{1}{\sigma}}\left(G\left(\boldsymbol{p}_{j>0},U\right)\right)^{\frac{\sigma(U)-1}{\sigma(U)}}.$$

Substituting the optimal choice of a_1 into (22) and simplifying gives

$$U=\left(\frac{w}{\left(\sum_{n'}\omega_{n'0}^{\gamma}U^{\epsilon_{n'}\gamma}p_{n'0}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}}\right)(B^p)^{\frac{1}{1-\sigma(U)}},$$

where $B^p=\sum_{n'}c_{n'0}p_{n'0}/w$ is the current consumption-to-wealth ratio. Hence the consumption-to-wealth ratio summarizes the future prices and returns relevant for the current self's welfare.

Now compare this household to one facing base prices and returns at τ_0. The money-metric wealth w_{τ_0} that leaves the household on the same indifference curve U satisfies

$$\left(\frac{w}{\left(\sum_{n'}\omega_{n'0}^{\gamma}U^{\epsilon_{n'}\gamma}p_{n'0}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}}\right)(B^p(U))^{\frac{1}{1-\sigma(U)}}=\left(\frac{w_{\tau_0}}{\left(\sum_{n'}\omega_{n'0}^{\gamma}U^{\epsilon_{n'}\gamma}p_{n'0\tau_0}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}}\right)\left(B_{\tau_0}^p(U)\right)^{\frac{1}{1-\sigma(U)}}.$$

Therefore,

$$w_{\tau_0}=w\left(\frac{\sum_{n'}\omega_{n'0}^{\gamma}U^{\epsilon_{n'}\gamma}p_{n'0\tau_0}^{1-\gamma}}{\sum_{n'}\omega_{n'0}^{\gamma}U^{\epsilon_{n'}\gamma}p_{n'0}^{1-\gamma}}\right)^{\frac{1}{1-\gamma}}\left(\frac{B^p(U)}{B_{\tau_0}^p(U)}\right)^{\frac{1}{1-\sigma(U)}}.$$

The first term is static inflation for a household on indifference curve U, and the second term is the change in the consumption-to-wealth ratio for that household. As in the model with commitment, the change in the consumption-to-wealth ratio is a sufficient statistic for how changes in future prices and returns affect the household in the present. Thus, given time separability, dynamic welfare is the same with or without commitment.

D.1.2 Leisure

Our baseline framework abstracts from labor-leisure choice. For rentiers, this assumption can be replaced by the assumption that rentiers choose to not work given their labor productivity. In this case, Proposition 4 applies without modification.[31]

We can allow non-rentiers to make labor-leisure choices and still use Proposition 5 to infer their welfare, if relative static budget shares only depend on utility and static prices of goods and services. This assumption rules out non-separabilities between consumption choices and leisure. For example, consider the utility function

$$U^{\frac{\sigma-1}{\sigma}} = \tilde{P}\left(\boldsymbol{c}\left(s^0\right), U\right)^{\frac{\sigma-1}{\sigma}} + \tilde{F}\left(\left\{\boldsymbol{c}(s^t)\right\}_{t>0}, \boldsymbol{\pi}, U\right)^{\frac{\sigma-1}{\sigma}} + \tilde{H}\left(\left\{\boldsymbol{l}(s^t)\right\}_{t\geq 0}, \boldsymbol{\pi}, U\right),$$

where $\tilde{P}$ and $\tilde{F}$ are aggregators over consumption of goods and services for the present and the future, and $\tilde{H}$ is an aggregator over leisure choices. These preferences have the useful feature that Lemma 1 is still valid and static relative budget shares depend only on static relative prices and U. This means that Proposition 5 can be used without modification. We do not pursue this extension in our empirical application.

D.1.3 Changes in Mortality

While the baseline model accommodates changes in mortality risk as a function of age, it assumes that mortality risk is fixed as a function of calendar time. Allowing for secular changes in mortality risk can alter the results because changes in welfare caused by changes in mortality risk are not necessarily reflected in changes of the consumption-to-wealth ratio.

To extend the results to allow for exogenous changes in mortality risk, consider the utility function

$$U^{\frac{\sigma-1}{\sigma}} = \lambda_P \tilde{P}\left(\boldsymbol{c}\left(s^0\right), U\right)^{\frac{\sigma-1}{\sigma}} + \lambda_P \lambda_F \beta \tilde{F}\left(\left\{\boldsymbol{c}(s^t)\right\}_{t>0}, \boldsymbol{\pi}, U\right)^{\frac{\sigma-1}{\sigma}} + [1-\lambda_P + \lambda_P(1-\lambda_F)\beta]\bar{c}^{\frac{\sigma-1}{\sigma}} U^{\frac{\sigma-1}{\sigma}+\varepsilon},$$

where $\tilde{P}$ and $\tilde{F}$ are present and future aggregators that are homogeneous of degree one in quantities (for a given U), parameters λ_P and λ_F are the probabilities of surviving in the present and the future, β is a discount factor, and σ is the constant EIS. Upon death, the household receives a consumption-equivalent payoff of $\bar{c}$ — the lower is $\bar{c}$, the higher is the

[31] If rentiers do earn risk-free labor income, as in the extension with risk-free cashflows, then we instead need to assume that hours do not vary as a function of calendar time (they do not have to be zero). This requires assuming that the number of hours worked by rentiers is fixed conditional on age (e.g. a nine-to-five job prior to retirement).

value of statistical life.[32] The parameter ε determines wealth effects for the value of life. All of these parameters may vary as a function of observable characteristics (principally, age).

The shadow expenditure function associated with these preferences is

$$e(\boldsymbol{q},\boldsymbol{\pi},\boldsymbol{\lambda},U)=\lambda_P^{\frac{\sigma}{1-\sigma}}\left(P(\boldsymbol{p}(s^0),U)^{1-\sigma}+\lambda_F^{\sigma}\beta^{\sigma}F(\left\{\boldsymbol{q}(s^t)\right\}_{t>0},\boldsymbol{\pi},U)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}\left[1-[1-\lambda_P+\lambda_P(1-\lambda_F)\beta]\bar{c}^{\frac{\sigma-1}{\sigma}}U^{\varepsilon}\right]^{\frac{\sigma}{\sigma-1}}U,$$

where P and F are the corresponding homogeneous price aggregators. For this problem, the lotteries the household faces are parameterized by the stochastic process over prices and returns, $\boldsymbol{\pi}$, as well as mortality risk, $\boldsymbol{\lambda}$. Whereas preferences are time-separable in consumption choices, $\boldsymbol{c}$, and probabilities over market outcomes, $\boldsymbol{\pi}$, they are not time-separable in mortality risk $\boldsymbol{\lambda}$.

As long as $\boldsymbol{\lambda}$ is constant as a function of calendar time, our results hold without modification. However, if $\boldsymbol{\lambda}$ changes over time, then time-separability is violated and our results cannot be used. Survival probabilities are not isomorphic to the other probabilities in the model because households cannot alter the pay-off upon death by saving.

To extend our results to this setting, we again index decision problems by calendar date, τ, which now also indexes survival probabilities, λ_P and λ_F. Following Schelling (1968), denote the value of statistical life for cohort τ to be the marginal willingness to pay for increasing survival probabilities in the present and the future relative to initial wealth w:

$$\Phi_P(\tau,w)=\frac{\partial\log m(\tau,w,\boldsymbol{y})}{\partial\log\lambda_P},\qquad\text{and}\qquad\Phi_F(\tau,w)=\frac{\partial\log m(\tau,w,\boldsymbol{y})}{\partial\log\lambda_F}.$$

The marginal willingness to pay to reduce the probability of death is an important input into cost-benefit analysis of policies that affect longevity.[33] If preferences are non-homothetic, then these elasticities depend on initial wealth.[34]

The extension of Proposition 4 to this environment is that the money-metric for rentiers,

[32]Some papers effectively assume that $\bar{c}^{(\sigma-1)/\sigma}=0$, but this is arbitrary. For example, it implies that the value of life switches from positive to negative as σ goes from above one to below one. Because $\bar{c}^{(\sigma-1)/\sigma}$ determines the household's willingness to pay to reduce the probability of death, its value should instead be disciplined by the value of statistical life (e.g. as in Jones and Klenow, 2016). Our formula in equation (23) directly uses information about the value of statistical life, instead of calibrating $\bar{c}$.

[33]See Ashenfelter (2006) for a discussion of how such statistics can be estimated.

[34]If the aggregators P and F are independent of U and $\varepsilon=0$, then the value of life relative to wealth does not vary as a function of wealth.

$m(\tau, w, \mathbf{0})$, solves the following fixed point problem:

$$\log m(\tau, w, \mathbf{0}) = \log w - \int_{\tau_0}^{\tau} \left(\sum_{n \in N} B_n(x, w_x^*) \frac{d \log p_n}{dx} - \frac{1}{1-\sigma} \frac{d \log B^P(x, w_x^*)}{dx} \right) dx \tag{23}$$
$$- \int_{\tau_0}^{\tau} \left(\Phi_P(x, w_x^*) \frac{d \log \lambda_P(x)}{dx} + \Phi_F(x, w_x^*) \frac{d \log \lambda_F(x)}{dx} \right) dx$$
$$+ \int_{\tau_0}^{\tau} \frac{\sigma}{1-\sigma} \left(1 - B^P(x, w_x^*) \right) \frac{d \log \lambda_F(x)}{dx} dx,$$

where $m(x, w_x^*, \mathbf{0}) = m(\tau, w, \mathbf{0})$ for each $x \in [\tau_0, \tau]$. The first line of (23) is identical to the one in Proposition 4. The remaining two lines adjust for changes in mortality risk. As expected, if mortality risk does not vary as a function of calendar date, $d \log \lambda_F / dx = d \log \lambda_P / dx = 0$, then (23) simplifies to the expression in Proposition 4.

The intuition for the new terms is the following. The second line of (23) adds the value of increased survival to the money-metric. This is very similar to how price changes must accounted for: we integrate the demand curve for the value of life with respect to changes in mortality risk. The subtlety is that, as with prices, we must integrate compensated demand curves rather than Marshallian demand curves. This implies that there is a fixed point term in the second line unless preferences are homothetic. The final line of (23) accounts for the fact that, holding all else fixed, changes in the future survival probability changes the consumption-wealth ratio. Since the second line is fully accounting for the welfare changes caused by changes in the future survival probability, the third line purges from $\frac{1}{1-\sigma} \frac{d \log B^P(x, w_x^*)}{dx}$ those changes caused by changes in future survival probability.

A simplification of (23) results if the utility payoff of death is zero: $\bar{c}^{\frac{\sigma-1}{\sigma}} = 0$. This is because, in this case, the marginal value of increasing survival probability in the future exactly offsets the adjustment in the consumption-wealth ratio:

$$\Phi_F(\tau, w) = \frac{\sigma}{1-\sigma} \left[1 - B^p(\tau, w) \right].$$

This means that (23) simplifies to

$$\log m(\tau, w) = \log w - \int_{\tau_0}^{\tau} \left(\sum_{n \in N} B_n(x, w_x^*) \frac{d \log p_n}{dx} - \frac{1}{1-\sigma} \frac{d \log B^P(x, w_x^*)}{dx} + \Phi_P(x, w_x^*) \frac{d \log \lambda_P(x)}{dx} \right) dx,$$

so that changes in the consumption-wealth ratio appropriately account for changes in welfare caused by changes in future survival probabilities. Although assuming $\bar{c}^{\frac{\sigma-1}{\sigma}} = 0$ is common in the literature and simplifies our results, there is no empirically compelling reason to treat this as a benchmark.

To recap, our results can be extended to account for changing mortality provided with additional information on the value of statistical life as a function of wealth and age. In our empirical application, we abstract from these issues and treat mortality risk as constant over time. We emphasize that this does not imply that we take mortality risk to be constant as a function of age. Variation in mortality rates due to changes in age are fully accounted for by our baseline result in Proposition 4. Rather, the assumption we make in our empirical results is that survival probabilities are constant as a function of time conditional on age, and we do not pursue this extension in our empirical illustration.

D.2 Non-Time-Separable Non-Homotheticity

Consider the preferences in (17). Since these preferences are additively separable, we use the Frisch elasticity of intertemporal substitution, instead of the compensated (Hicksian) elasticity of substitution to define our sufficient statistics. The *Frisch* elasticity of intertemporal substitution (EIS) controls how spending responds to a uniform change in prices, holding the marginal utility of wealth constant. In comparison, the compensated elasticity holds utility constant.

We start with a preliminary result characterizing the Frisch EIS for this class of preferences. The Frisch elsaticity of intertemporal substitution, in period t, for the preferences above is

$$\sigma_t(\tau, w, \boldsymbol{y}) = \left[(1-\gamma)\frac{Var_{B(s^t)}(\epsilon_n)}{\mathbb{E}_{B(s^t)}\left[\epsilon_n\right]^2} + 1 - \frac{\left[1-\frac{1}{\theta}\right]}{\mathbb{E}_{B(s^t)}\left[\epsilon_n\right]}\right]^{-1}. \tag{24}$$

where the variance and expectation use period t budget shares, denoted $B(s^t)$, as weights.[35] Since budget shares vary as a function of $(\tau, w, \boldsymbol{y})$, the Frisch EIS also varies as a function of these variables. The Frisch EIS for the first period, period 0, is denoted by $\sigma_0(\tau, w, \boldsymbol{y})$.

The Frisch EIS is not, in general, equal to the usual parameter θ in (17) because the curvature induced by the ϵ_n's interacts with intertemporal choices. Note that, in the special case of homothetic preferences, where $\epsilon_n = 1$ for every n, we recover the usual fact that the Frisch EIS, $\sigma_0(\tau, w, \boldsymbol{y})$, is identically equal to θ.

Now, we show how our results are altered for rentiers. For simplicity, we assume rentiers face complete markets, though the result can be extended to allow for incomplete markets along similar lines as in Proposition 4.

Proposition 7. *The money-metric for rentiers with preferences* (17) *satisfies the following*

[35] The *Frisch* elasticity of intertemporal substitution (EIS) controls how spending responds to a uniform change in prices, holding the marginal utility of wealth constant.

$$\log m(\tau, w, \mathbf{0}) \;=\; \log w - \int_{\tau_0}^{\tau} \left(\sum_{n \in N} B_n(x, w_x^*, \mathbf{0}) \frac{d \log p_n}{dx} - \frac{d \log B^P(x, w_x^*, \mathbf{0})/dx}{1 - \sigma_0(x, w_x^*, \mathbf{0})} \right) + error\, dx.$$

where w_x^* solves the equation $m(x, w_x^*, \mathbf{0}) = m(\tau, w, \mathbf{0})$ for each $x \in [\tau_0, \tau]$. The boundary condition is that $m(\tau_0, w, \mathbf{0}) = w$. The error term shrinks to zero as $\sigma_0 \to 0$, which happens as $\theta \to 0$. The error term is

$$error = -\sigma_0 \left[(1-\gamma) \left(Cov_{B_0}\left(\frac{\epsilon_i}{\mathbb{E}_{B_0}[\epsilon_i]}, d\log p_{0i}\right) - \mathbb{E}^{NPV}_{\sigma_t} \left[Cov_{B_t}\left(\frac{\epsilon_i}{\mathbb{E}_B[\epsilon_i]}, d\log p_{ti}\right) \right] \right) + Cov^{NPV}\left(\frac{\sigma_t}{\bar{\sigma}}, \mathbb{E}_{B_t}\left[d \log \frac{p_t}{1+r_t} \right] \right) \right].$$

Proposition 7 is the same as Proposition 4 except that there is an error term (which varies by τ and w). The most important observation is that, since the error term is multiplied by σ_0, as the EIS tends to zero, the error also tends to zero. Moreover, even disregarding the fact that the error term scales in σ_0, the remaining terms are likely to be quantitatively small in practice.

The first component of the error term depends on the difference between a covariance in the present and the net-present value of future covariances weighted by the Frisch EIS in those periods ($\mathbb{E}^{NPV}_{\sigma_t}$ denotes a net-present value average weighted by σ_t). These are the covariances between the slope of Engel curves, disciplined by $\epsilon_n/\mathbb{E}_B(\epsilon_n)$, and the change in prices, given by $d\log p_n(s^j)/dx$. Hence, if the covariance of price shocks with the slope of the Engel curves is roughly constant, this term is small.

The second component of the error term depends on the covariance of variations in the Frisch EIS (measured by $\sigma_t/\bar{\sigma}$, where $\bar{\sigma}$ is net-present value average of all σ's) with price and return shocks. This term too is likely to be small in practice, since empirically, the EIS does not vary much (Best et al., 2020). Both of these error terms are multiplied by the Frisch EIS in the present, σ_0, which is also likely to be small.

In our Monte Carlo exercises, where $\sigma_0 \approx 0.1$, we find that these error terms are extremely small even for very large shocks.

The next proposition shows that Proposition 5 also applies to non-rentiers with Comin et al. (2021) preferences as the Frisch EIS tends to zero.

Proposition 8. *Proposition 5 applies to non-rentiers with preferences defined in Equation* (17) *when $\sigma_0 = 0$ (which happens if, for example, $\theta \to 0$).*

Intuitively, when the first period Frisch EIS satisfies $\sigma_0 = 0$, preferences become Leontief across periods and so $C_j = C$ for all j and the common value C is a monotone function of lifetime utility. We leave the proofs and additional discussion in Section D.

Proof of Proposition 7. For simplicity, we assume that rentiers solve a complete markets problem. (The results could be extended along similar lines as the main paper to incomplete markets.) The intertemporal expenditure function for rentiers is defined by the following constrained optimization problem:

$$\min_{E_t} \sum_{t=0}^{\infty} \frac{E_t}{(1+r_t)}$$

such that

$$U = \left[\sum_t \beta^t C_t^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}},$$

and expenditures in period t satisfy

$$E_t = \left[\sum_i \omega_{ti} \left[C_t^{\epsilon_i} p_{ti} \right]^{1-\gamma} \right]^{\frac{1}{1-\gamma}}.$$

The Lagrangian is

$$\sum_{t=0}^{\infty} \frac{E_t}{(1+r_t)} - \lambda \left[U - \left[\sum_t \beta^t C_t^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}} \right] + \sum_t \mu_t \left[E_t - \left[\sum_i \omega_{ti} \left[C_t^{\epsilon_i} p_{ti} \right]^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \right].$$

The first order conditions are

$$\frac{1}{(1+r_t)} = \mu_t,$$

and

$$\lambda \frac{\theta}{\theta-1} \left[\sum_t \beta^t C_t^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}-1} \beta^t \frac{\theta-1}{\theta} C_t^{\frac{\theta-1}{\theta}-1} = \mu_t \frac{1}{1-\gamma} \left[\sum_i \omega_{ti} \left[C_t^{\epsilon_i} p_{ti} \right]^{1-\gamma} \right]^{\frac{1}{1-\gamma}-1} \sum_i \omega_{ti} \left[C_t^{\epsilon_i} p_{ti} \right]^{1-\gamma} (1-\gamma) \epsilon_i \frac{1}{C_t}.$$

Combine these first order condtions to get the equation,

$$\frac{1}{(1+r_t)} E_t^{\gamma} \sum_i \omega_{ti} \left[C_t^{\epsilon_i} p_{ti} \right]^{1-\gamma} \epsilon_i \frac{1}{C_t} = \lambda U^{\frac{1}{\theta}} \beta^t C_t^{\frac{-1}{\theta}}$$

or

$$\frac{1}{(1+r_t)} E_t^{\gamma} \left[\sum_i \omega_{ti} \left[C_t^{\epsilon_i} p_{ti} \right]^{1-\gamma} \right] \sum_i \frac{\omega_{ti} \left[C_t^{\epsilon_i} p_{ti} \right]^{1-\gamma}}{\sum_i \omega_{ti} \left[C_t^{\epsilon_i} p_{ti} \right]^{1-\gamma}} \epsilon_i \frac{1}{C_t} = \lambda U^{\frac{1}{\theta}} \beta^t C_t^{\frac{-1}{\theta}}.$$

Using the fact that budget shares are:

$$b_{ti} = \frac{\partial \log E_t}{\partial \log p_{ti}} = \frac{\omega_{ti}\left[C_t^{\epsilon_i} p_{ti}\right]^{1-\gamma}}{\sum_i \omega_{ti}\left[C_t^{\epsilon_i} p_{ti}\right]^{1-\gamma}},$$

we can rewrite the first-order condition as:

$$\frac{1}{(1+r_t)}\frac{E_t}{C_t}\sum_i b_{ti}\epsilon_i = \lambda U^{\frac{1}{\theta}}\beta^t C_t^{\frac{-1}{\theta}}.$$

The log-linearized demand system is

$$-d\log(1+r_t) + d\log E_t - d\log C_t + \frac{Cov_b(\epsilon_i, d\log b_{ti})}{\mathbb{E}_b\left[\epsilon_i\right]} = d\log\lambda + \frac{1}{\theta}d\log U - \frac{1}{\theta}d\log C_t,$$

or

$$\left[1-\frac{1}{\theta}\right]d\log C_t - d\log E_t = -d\log\lambda - \frac{1}{\theta}d\log U - d\log(1+r_t) + \frac{Cov_b(\epsilon_i, d\log b_{ti})}{\mathbb{E}_b\left[\epsilon_i\right]},$$

where

$$d\log E_t = \mathbb{E}_{b_t}\left[d\log p_{ti}\right] + \mathbb{E}_{b_t}\left[\epsilon_i\right] d\log C_t,$$

and

$$d\log b_{ti} = (1-\gamma)\left[d\log p_{ti} - d\log E_t\right] + (1-\gamma)\epsilon_i d\log C_t$$

To keep the household on the same indifference curve, we also require that

$$\sum_t \frac{E_t/(1+r)^t}{\sum_{t'} E_{t'}/(1+r)^{t'}}\left(d\log E_t - \mathbb{E}_{b_t}\left[d\log p_t\right]\right) = 0,$$

which is a consequence of Shephard's lemma. Hence, more generally, for any compensated variation, the following equations must hold:

$$\left[1-\frac{1}{\theta}\right]d\log C_t - d\log E_t = -d\log\lambda - d\log(1+r_t) + \frac{Cov_b(\epsilon_i, d\log b_{ti})}{\mathbb{E}_b\left[\epsilon_i\right]},$$

$$\frac{1}{\mathbb{E}_{b_t}\left[\epsilon_i\right]}\left[d\log E_t - \mathbb{E}_{b_t}\left[d\log p_{ti}\right]\right] = d\log C_t,$$

$$d\log b_{ti} = (1-\gamma)\left[d\log p_{ti} - d\log E_t\right] + (1-\gamma)\epsilon_i d\log C_t,$$

$$\sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \left(d\log E_t - \mathbb{E}_{b_t}\left[d\log p_t\right]\right) = 0.$$

Combine these equations to get

$$\frac{\left[1-\frac{1}{\theta}\right]}{\mathbb{E}_{b_t}\left[\epsilon_i\right]}\left[d\log E_t - \mathbb{E}_{b_t}\left[d\log p_{ti}\right]\right] - d\log E_t = -d\log\lambda - d\log(1+r_t) + \frac{Cov_b(\epsilon_i, d\log b_{ti})}{\mathbb{E}_b\left[\epsilon_i\right]},$$

$$\begin{aligned} d\log b_{ti} = &-(1-\gamma)\left[d\log E_t - d\log p_{ti}\right] \\ &+ (1-\gamma)\frac{\epsilon_i}{\mathbb{E}_{b_t}\left[\epsilon_i\right]}\left[d\log E_t - \mathbb{E}_{b_t}\left[d\log p_{ti}\right]\right], \end{aligned}$$

and

$$\sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \left(d\log E_t - \mathbb{E}_{b_t}\left[d\log p_t\right]\right) = 0.$$

Or

$$\begin{aligned} \frac{\left[1-\frac{1}{\theta}\right]}{\mathbb{E}_{b_t}\left[\epsilon_i\right]}\left[d\log E_t - \mathbb{E}_{b_t}\left[d\log p_{ti}\right]\right] - d\log E_t = &-d\log\lambda - d\log(1+r_t) + \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]}Cov_b(\epsilon_i, d\log p_{ti}) \\ &+ (1-\gamma)\frac{Var_b(\epsilon_i)}{\mathbb{E}_b\left[\epsilon_i\right]^2}\left[d\log E_t - \mathbb{E}_{b_t}\left[d\log p_{ti}\right]\right], \end{aligned}$$

and

$$\sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \left(d\log E_t - \mathbb{E}_{b_t}\left[d\log p_t\right]\right) = 0.$$

Define the inverse Frisch elasticity $\kappa_t = \left[\frac{\left[1-\frac{1}{\theta}\right]}{\mathbb{E}_{b_t}[\epsilon_i]} - (1-\gamma)\frac{Var_b(\epsilon_i)}{\mathbb{E}_b[\epsilon_i]^2}\right]$. Then we can write

$$\kappa_t\left[d\log E_t - \mathbb{E}_{b_t}\left[d\log p_{ti}\right]\right] = d\log E_t - d\log\lambda - d\log(1+r_t) + \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]}Cov_b(\epsilon_i, d\log p_{ti}),$$

and

$$\sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \left(d\log E_t - \mathbb{E}_{b_t}\left[d\log p_t\right]\right) = 0.$$

Substituting one equation into the second implies that

$$\sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})}\kappa_t^{-1}\left(d\log E_t - d\log\lambda - d\log(1+r_t) + \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]}Cov_b(\epsilon_i, d\log p_{ti})\right) = 0.$$

Denote by $\bar{\kappa} = \left[\sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \kappa_t^{-1}\right]^{-1}$ the harmonic net present value weighted average of κ_t. Then we can rewrite the previous equation as

$$\bar{\kappa} \sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \kappa_t^{-1} \left(d\log E_t - d\log(1+r_t) + \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]} Cov_b(\epsilon_i, d\log p_{ti}) \right) = d\log\lambda,$$

$$\mathbb{E}_{\kappa}^{NPV}\left[\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t) + \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]} Cov_b(\epsilon_i, d\log p_{ti})\right] = d\log\lambda,$$

where $\mathbb{E}_{\kappa^{-1}}^{NPV}$ is an average that uses weights $\frac{\frac{E_t/(1+r_t)}{\sum_{t''} E_{t''}/(1+r_{t''})}\kappa_t^{-1}}{\left[\sum_{t'} \frac{E_{t'}/(1+r_{t'})}{\sum_{t'} E_{t'}/(1+r_{t''})}\kappa_{t'}^{-1}\right]}$, which is a net-present value weighted κ_t. Hence,

$$\begin{aligned}\left[\kappa_t - 1\right] d\log E_t - \left[\kappa_t - 1\right] \mathbb{E}_{b_t}\left[d\log p_{ti}\right] = {} & \mathbb{E}_{b_t}\left[d\log p_{ti}\right] - d\log(1+r_t) + \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]} Cov_b(\epsilon_i, d\log p_{ti}) \\ & - \mathbb{E}_{\kappa^{-1}}^{NPV}\left[\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t) + \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]} Cov_b(\epsilon_i, d\log p_{ti})\right]\end{aligned}$$

Evaluate this for $t = 0$ to get

$$\begin{aligned}\left[\kappa_0 - 1\right]\left[d\log E_0 - \mathbb{E}_{b_0}\left[d\log p_{0i}\right]\right] = {} & \mathbb{E}_{b_0}\left[d\log p_{0i}\right] + \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]} Cov_b(\epsilon_i, d\log p_{ti}) \\ & - \mathbb{E}_{\kappa^{-1}}^{NPV}\left[\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t) + \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]} Cov_b(\epsilon_i, d\log p_{ti})\right],\end{aligned}$$

or

$$\begin{aligned}d\log E_0 = {} & \left[\kappa_0 - 1\right]^{-1} \kappa_0 \mathbb{E}_{b_0}\left[d\log p_{0i}\right] + \left[\kappa_0 - 1\right]^{-1} \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]} Cov_b(\epsilon_i, d\log p_{ti}) \\ & - \left[\kappa_0 - 1\right]^{-1} \mathbb{E}_{\kappa^{-1}}^{NPV}\left[\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t) + \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]} Cov_b(\epsilon_i, d\log p_{ti})\right].\end{aligned}$$

Let W denote wealth, and note that the consumption-to-wealth ratio satisfies

$$\begin{aligned}d\log E_0 - d\log W = {} & \left[\kappa_0 - 1\right]^{-1} \kappa_0 \mathbb{E}_{b_0}\left[d\log p_{0i}\right] + \left[\kappa_0 - 1\right]^{-1} \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]} Cov_b(\epsilon_i, d\log p_{ti}) \\ & - \left[\kappa_0 - 1\right]^{-1} \mathbb{E}_{\kappa^{-1}}^{NPV}\left[\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t) + \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]} Cov_b(\epsilon_i, d\log p_{ti})\right] \\ & - \sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \left(\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t)\right),\end{aligned}$$

where the last line uses the fact that $d\log W = \sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \left(\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t)\right)$. This fact is a consequence of two equations: (1) the accounting identity $d\log W = \sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \left(d\log E_t - d\log(1+r_t)\right)$, and (2) the fact that any compensated variation must satisfy $\sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \left(d\log E_t - \mathbb{E}_{b_t}\left[d\log p_t\right]\right) = 0$. Combining these two equations yields the desired result.

$$
\begin{aligned}
d\log E_0 - d\log W = {} & \left[\kappa_0 - 1\right]^{-1} \kappa_0 \mathbb{E}_{b_0}\left[d\log p_{0i}\right] + \left[\kappa_0 - 1\right]^{-1} \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]} Cov_b(\epsilon_i, d\log p_{ti}) \\
& - \left[\kappa_0 - 1\right]^{-1} \mathbb{E}^{NPV}_{\kappa^{-1}}\left[\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t) + \frac{(1-\gamma)}{\mathbb{E}_b\left[\epsilon_i\right]} Cov_b(\epsilon_i, d\log p_{ti})\right] \\
& - \sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \left(\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t)\right),
\end{aligned}
$$

or

$$
\begin{aligned}
d\log E_0 - d\log W = {} & \left[\kappa_0 - 1\right]^{-1} \kappa_0 \mathbb{E}_{b_0}\left[d\log p_{0i}\right] \\
& + \frac{(1-\gamma)}{\kappa_0 - 1}\left[Cov_b\left(\frac{\epsilon_i}{\mathbb{E}_b\left[\epsilon_i\right]}, d\log p_{ti}\right) - \mathbb{E}^{NPV}_{\kappa^{-1}}\left[Cov_b\left(\frac{\epsilon_i}{\mathbb{E}_b\left[\epsilon_i\right]}, d\log p_{ti}\right)\right]\right] \\
& - \frac{\bar{\kappa}}{\kappa_0 - 1} \sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \left[\frac{1}{\kappa_t} + \frac{\left[\kappa_0 - 1\right]}{\bar{\kappa}}\right] \left(\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t)\right)
\end{aligned}
$$

where $\bar{\kappa} = \left[\sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \kappa_t^{-1}\right]^{-1}$ is the harmonic net present value weighted average of κ_t. Then we can write

$$
\begin{aligned}
d\log E_0 - d\log W = {} & \frac{\kappa_0}{\kappa_0 - 1} \mathbb{E}_{b_0}\left[d\log p_{0i}\right] \\
& - \frac{\kappa_0}{\kappa_0 - 1} \sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \left[\frac{1}{\kappa_0}\frac{\bar{\kappa}}{\kappa_t} + \left[1 - \frac{1}{\kappa_0}\right]\right] \left(\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t)\right) \\
& + \frac{(1-\gamma)}{\kappa_0 - 1}\left[Cov_b\left(\frac{\epsilon_i}{\mathbb{E}_b\left[\epsilon_i\right]}, d\log p_{ti}\right) - \mathbb{E}^{NPV}_{\kappa^{-1}}\left[Cov_b\left(\frac{\epsilon_i}{\mathbb{E}_b\left[\epsilon_i\right]}, d\log p_{ti}\right)\right]\right],
\end{aligned}
$$

or

$$
\begin{aligned}
d\log E_0 - d\log W = {} & \frac{\kappa_0}{\kappa_0 - 1} \mathbb{E}_{b_0}\left[d\log p_{0i}\right] - \frac{\kappa_0}{\kappa_0 - 1} \sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \left(\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t)\right) \\
& - \frac{1}{\kappa_0 - 1} \sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})} \left[\frac{\bar{\kappa} - \kappa_t}{\kappa_t}\right] \left(\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t)\right) \\
& + \frac{(1-\gamma)}{\kappa_0 - 1}\left[Cov_b\left(\frac{\epsilon_i}{\mathbb{E}_b\left[\epsilon_i\right]}, d\log p_{ti}\right) - \mathbb{E}^{NPV}_{\kappa^{-1}}\left[Cov_b\left(\frac{\epsilon_i}{\mathbb{E}_b\left[\epsilon_i\right]}, d\log p_{ti}\right)\right]\right],
\end{aligned}
$$

Therefore,

$$\frac{\kappa_0 - 1}{\kappa_0}\left[d\log E_0 - d\log W\right] = \mathbb{E}_{b_0}\left[d\log p_{0i}\right] - \sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})}\left(\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t)\right)$$
$$- \frac{1}{\kappa_0}\sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})}\left[\frac{\bar{\kappa}-\kappa_t}{\kappa_t}\right]\left(\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t)\right)$$
$$+ \frac{(1-\gamma)}{\kappa_0}\left[Cov_b\left(\frac{\epsilon_i}{\mathbb{E}_b\left[\epsilon_i\right]}, d\log p_{ti}\right) - \mathbb{E}^{NPV}_{\kappa^{-1}}\left[Cov_b\left(\frac{\epsilon_i}{\mathbb{E}_b\left[\epsilon_i\right]}, d\log p_{ti}\right)\right]\right],$$

or

$$\sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})}\left(\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t)\right) = \mathbb{E}_{b_0}\left[d\log p_{0i}\right] - \frac{\kappa_0-1}{\kappa_0}\left[d\log E_0 - d\log W\right]$$
$$+ \frac{(1-\gamma)}{\kappa_0}\left[Cov_b\left(\frac{\epsilon_i}{\mathbb{E}_b\left[\epsilon_i\right]}, d\log p_{ti}\right) - \mathbb{E}^{NPV}_{\kappa^{-1}}\left[Cov_b\left(\frac{\epsilon_i}{\mathbb{E}_b\left[\epsilon_i\right]}, d\log p_{ti}\right)\right]\right]$$
$$- \frac{1}{\kappa_0}\sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})}\left[\frac{\bar{\kappa}-\kappa_t}{\kappa_t}\right]\left(\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t)\right).$$

Hence, from Shephard's lemma, the change in money metric is in the change in wealth deflated by average prices:

$$d\log m = d\log W - \sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})}\left(\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t)\right)$$
$$= d\log W - \mathbb{E}_{b_0}\left[d\log p_{0i}\right] + \left[1 - \frac{1}{\kappa_0}\right]\left[d\log\frac{E_0}{W}\right]$$
$$- \frac{(1-\gamma)}{\kappa_0}\left[Cov_{b_0}\left(\frac{\epsilon_i}{\mathbb{E}_{b_0}\left[\epsilon_i\right]}, d\log p_{0i}\right) - \mathbb{E}^{NPV}_{\kappa^{-1}}\left[Cov_b\left(\frac{\epsilon_i}{\mathbb{E}_b\left[\epsilon_i\right]}, d\log p_{ti}\right)\right]\right]$$
$$+ \frac{1}{\kappa_0}\sum_t \frac{E_t/(1+r_t)}{\sum_{t'} E_{t'}/(1+r_{t'})}\left[\frac{\bar{\kappa}-\kappa_t}{\kappa_t}\right]\left(\mathbb{E}_{b_t}\left[d\log p_t\right] - d\log(1+r_t)\right).$$

Integrating this expression, using the fundamental theorem of calculus, yields the desired result. Note that as $\theta \to 0$,we have

$$\lim_{\theta\to 0}\kappa_0 = \lim_{\theta\to 0}\left[\frac{\left[1-\frac{1}{\theta}\right]}{\mathbb{E}_{b_t}\left[\epsilon_i\right]} - (1-\gamma)\frac{Var_b(\epsilon_i)}{\mathbb{E}_b\left[\epsilon_i\right]^2}\right] = -\infty.$$

Hence, in this limit, the error term vanishes:

$$\lim_{\theta\to 0}\log m = \log W - \int\left(\mathbb{E}_{b_0}\left[d\log p_{0i}\right] - \left[d\log\frac{E_0}{W}\right]\right).$$

Proof of Proposition 8. Now consider the non-rentiers, who face a similar problem except receive risky labor income and are subject to borrowing constraints (as usual, borrowing constraints cannot bind for rentiers, so we can ignore them in the previous calculation). Non-rentiers maximize utility

$$\max U = \mathbb{E}\left[\sum_t \beta^t C_t^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}}$$

$$E_t = \left[\sum_i \omega_{ti}\left[C_t^{\epsilon_i} p_{ti}\right]^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

$$E_t(s) + \sum_k a_{kt}(s) = \sum_k R_{k,t}(s) a_{kt-1}(s) + y_t(s)$$

$$\sum_k a_{kt}(s) \leq -B_t(s).$$

Note that as $\theta \to 0$, the utility function converges to $U \to \max\left[\min_{t\geq 0} C_t\right]$. Furthermore, it is easy to see that for this problem,

$$\max\left[\min_{t\geq 0} C_t\right] = C_0.$$

This can be proved by contradiction. Suppose that a feasible consumption plan obtains its minimum at some history $s^{t^*} > 0$. The agent can reduce consumption in 0, increase savings, and increase $C(s^{t^*})$,which raises utility. Hence, the postulated consumption plan cannot be optimal. This argument holds as long as rates of return are positive, so that savings at time 0 can raise consumption in history s^{t^*}(the presence of a risk-free asset guarantees this time).

E Appendix for Section 5: Performance in Simulated Data

E.1 Estimation of PSID Income Groups

This appendix describes how we estimate the latent household income groups used to discipline the simulated income processes in Section 5. These estimates are used only to calibrate the Monte Carlo economies; they are not inputs into the empirical welfare estimates in Section 6.

Data and sample. The six non-rentier income groups are estimated from the PSID family interview. The unit of observation is the household-year, and households are followed over time using the identifier of the household head, that is, the PSID reference person. Household labor income is defined simply as the labor earnings of the head plus the labor earnings of the spouse. We restrict the sample to households with head age 20–64, require at least five retained observations per household, and estimate the process on observations with positive household labor income. The last restriction is imposed because the maintained stochastic process is specified in logs and does not include a separate nonemployment state. The retained sample used for estimation of parameters contains 115,008 household-year observations for 7,817 households and spans income years 1967–2018. Nominal income is converted to 2019 dollars using annual CPI-U.

Outcome used in estimation. For household i in year t, let Y_{it}^{H} denote annual household labor income and let P_t denote the CPI-U deflator to 2019 dollars. We construct real household labor income as

$$Y_{it}^{R} = \frac{Y_{it}^{H}}{P_t}.$$

The object taken to the income-process estimation is

$$\tilde{y}_{it} = \log Y_{it}^{R} - \lambda_t,$$

where λ_t is the common calendar-year effect in year t, computed as the sample mean of log real household labor income in that year. This removes aggregate movements common to all households, so the fitted process captures idiosyncratic income dynamics rather than inflation or economy-wide wage shifts.

Statistical model. Conditional on belonging to latent type $g \in \{1, \ldots, 6\}$, household income follows

$$\tilde{y}_{it} = \beta_{0g} + \beta_{1g}(a_{it} - 40) + \beta_{2g}(a_{it} - 40)^2 + z_{it} + \varepsilon_{it},$$

where a_{it} is the age of the household head, z_{it} is the persistent component of income, and ε_{it} is a transitory disturbance. The persistent component evolves as

$$z_{i,t+d} = \rho_g^d z_{it} + u_{i,t,t+d}, \qquad u_{i,t,t+d} \sim N\left(0,\, \sigma_{\eta,g}^2 \frac{1-\rho_g^{2d}}{1-\rho_g^2}\right),$$

and the transitory component is

$$\varepsilon_{it} \sim N(0, \sigma_{e,g}^2).$$

Each type therefore has its own age profile $(\beta_{0g}, \beta_{1g}, \beta_{2g})$, persistence ρ_g, persistent-shock standard deviation $\sigma_{\eta,g}$, transitory standard deviation $\sigma_{e,g}$, and population share π_g, with $\sum_{g=1}^{6} \pi_g = 1$. The dependence on the gap length d is important because the PSID becomes biennial after 1997 and because some households are missing in intermediate years. Relative to a one-type specification, the finite mixture allows the data to sort households into groups that differ in average income levels, life-cycle income profiles, persistence, and volatility.

Estimation procedure. Our estimation procedure follows the Kalman-filter/EM logic in Braxton et al. (2025), adapted here to allow for six latent income types. Let $\theta_g = (\beta_{0g}, \beta_{1g}, \beta_{2g}, \rho_g, \sigma_{\eta,g}, \sigma_{e,g})$ denote the parameter vector for type g. For household i, the likelihood contribution is

$$L_i(\Theta) = \sum_{g=1}^{6} \pi_g f_g(\tilde{y}_i \mid a_i; \theta_g),$$

where $f_g(\cdot)$ is the Gaussian state-space likelihood implied by the type-g model. The persistent state is initialized at its stationary variance, and the transition equation is adjusted for the exact time gap between successive retained observations. This matters because the panel is unbalanced and because spacing is not always one year.

The model is estimated by maximum likelihood on the household panel. Computationally, we proceed in stages. We first estimate the one-type model, which provides a benchmark fit and a stable starting point. We then initialize the six-type model by perturbing the one-type estimates and run a multiple-start expectation-maximization algorithm. In the E-step, each household is assigned posterior probabilities of belonging to each of the six types. In the M-step, each type-specific parameter vector is re-optimized using those posterior weights. Because mixture models can converge to poor local optima, we run several random starts and keep the solution with the highest observed log likelihood. After estimation, we relabel the six groups by fitted mean income at age 40 so that type labels are economically ordered and comparable across runs.

Mapping the PSID fit into the model. The estimated statistical model is richer than the income process used in the quantitative model. In particular, the PSID fit contains an age profile and a transitory component, whereas the quantitative model only needs an age-less AR(1) for the persistent component of labor income. We therefore map the statistical estimates into the simpler representation

$$\log y_t = \mu_g + z_t, \qquad z_{t+1} = \rho_g z_t + \eta_{t+1}, \qquad \eta_{t+1} \sim N(0, \sigma^2_{\eta,g}).$$

For each latent type g, the mean μ_g is computed as the fitted average log income over the empirical age distribution in the estimation sample, and then centered so that the mixture-weighted mean is zero:

$$\sum_{g=1}^{6} \pi_g \mu_g = 0.$$

This produces, for each type, the four objects used in the model: the type share π_g, mean log income μ_g, persistence ρ_g, and persistent-shock standard deviation $\sigma_{\eta,g}$. The transitory component $\sigma_{e,g}$ is not carried directly into the model.

Since the quantitative model is quarterly, while the PSID fit is annual, we convert the annual AR(1) parameters into quarterly ones using the standard time-aggregation formulas

$$\rho_{g,q} = \rho_{g,a}^{1/4}, \qquad \sigma_{\eta,g,q} = \sigma_{\eta,g,a} \sqrt{\frac{1-\rho_{g,q}^2}{1-\rho_{g,a}^2}}.$$

The mean μ_g and the type share π_g are unchanged by this conversion. Table 2 reports the final quarterly parameters used in the quantitative model.

Table 2: Estimated six-type household labor-income process used in the quantitative model

Type g	Mean μ_g	ρ_q	$\sigma_{\eta,q}$
1	-1.23	0.795	0.25
2	-0.46	0.966	0.29
3	0.03	0.987	0.11
4	0.32	0.996	0.05
5	0.07	0.979	0.17
6	0.35	0.988	0.07

Notes: The table reports the quarterly AR(1) parameters passed to the Monte Carlo solver. Means are centered so that the mixture-weighted mean of log income is zero. The richer PSID fit also estimates type-specific age-profile coefficients, type shares, and transitory standard deviations, but those are not carried directly into the quantitative model.

E.2 Monte Carlo Price-Shock Exercise

The price-shock exercise in Figure 3 is implemented as follows. At date $\tau = 1$, goods 1 and 2 are hit by unexpected price shocks: on impact, the price of good 1 decreases by 20% and the price of good 2 decreases by 10%. The price of good 3 is unchanged. From that date onward, period-by-period price changes gradually return to zero, while price levels continue adjusting until they reach a lower terminal steady-state vector.

To compute the exact welfare change from the price shock, we proceed as follows. For each household in the ergodic distribution of the pre-shock economy, we hold fixed the realized $\tau = 1$ wealth and income state, (w, y), and evaluate true dynamic money-metric utility at $\tau = 1$ with and without the price shock. The welfare change for household i due to the price shock at $\tau = 1$ is therefore

$$\Delta \log m_i^{\text{exact}} = \log m_{i,\tau}^{\text{shock}} - \log m_{i,\tau}^{\text{noshock}}.$$

To apply the sufficient-statistic method in Proposition 4, we first construct a two-period sample using data simulated from the model for the $\tau = 0$ and $\tau = 1$ cohorts. We record pre-shock expenditures at date $\tau = 0$ and at date $\tau = 1$, with and without the price shock, holding fixed the income states at $\tau = 1$. We define contaminated rentiers as households whose financial-wealth share exceeds 0.9 at date $\tau = 0$. On this subsample, we estimate Engel curves for the log expenditure-to-wealth ratio and for log budget shares as functions of log total wealth, at $\tau = 0$ and at $\tau = 1$ separately with and without the price shock.

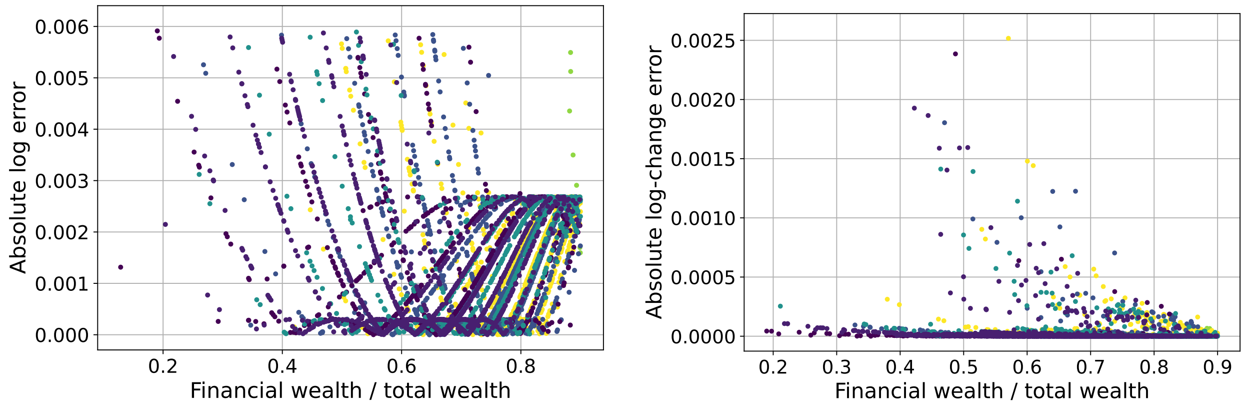
For the contaminated-rentier estimator, these fitted objects, together with the observed price vectors at $\tau = 0$ and $\tau = 1$, are fed into the money-metric procedure in Proposition 4 to recover estimated money-metric values before and after the shock. Denoting these estimates by $\hat{m}_{i,\tau}^{\text{shock}}$ and $\hat{m}_{i,\tau}^{\text{noshock}}$, the estimated welfare effect of the price shock at $\tau = 1$ for household i is

$$\widehat{\Delta \log m}_i = \log \hat{m}_{i,\tau}^{\text{shock}} - \log \hat{m}_{i,\tau}^{\text{noshock}}.$$

As in our empirical results, we report only households whose budget shares lie inside the convex hull of the contaminated-rentier budget shares used in the $\tau = 0$ estimation.

E.3 Time-Separable Monte Carlo with EIS Close to One

This appendix repeats the contaminated-rentier Monte Carlo exercises from Section 5 with an EIS close to one. We use the same calibration as in the benchmark time-separable exercise, but set $\sigma = 0.91$ rather than $\sigma = 0.1$, and the annual interest rate is 4%. This exercise shows that the method does not rely on a low EIS for accuracy when preferences are time separable; what matters is that the researcher uses the correct EIS.



(a) Level errors.

(b) Change errors.

Figure 13: Absolute log errors for our method using contaminated rentiers, with EIS close to one.

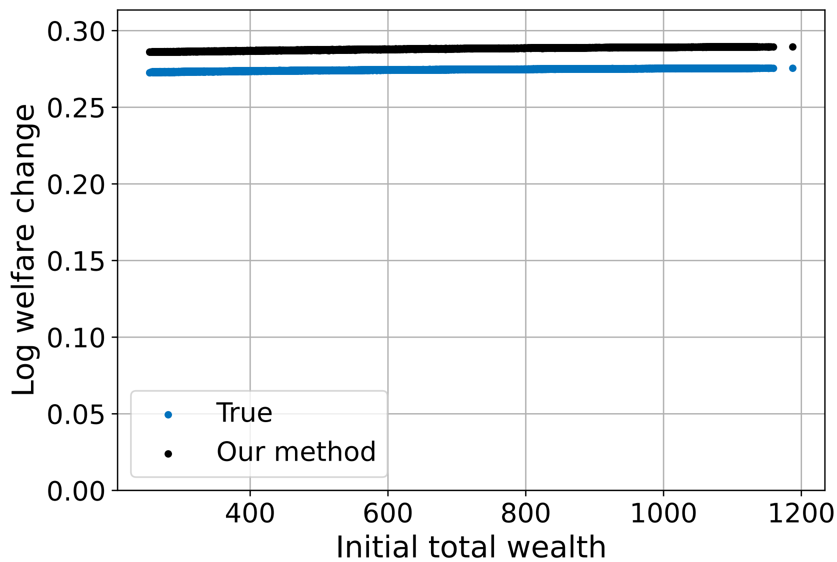


Figure 14: True welfare changes and estimated welfare changes under the unexpected price transition with EIS close to one, using contaminated rentiers.

E.4 Frisch EIS under Non-Time-Separable Preferences

Figure 15 plots the first-period Frisch EIS, σ_0, for simulated non-rentiers in the Comin et al. Monte Carlo exercise. This is the EIS that enters the sufficient-statistic formula when preferences are not time separable.

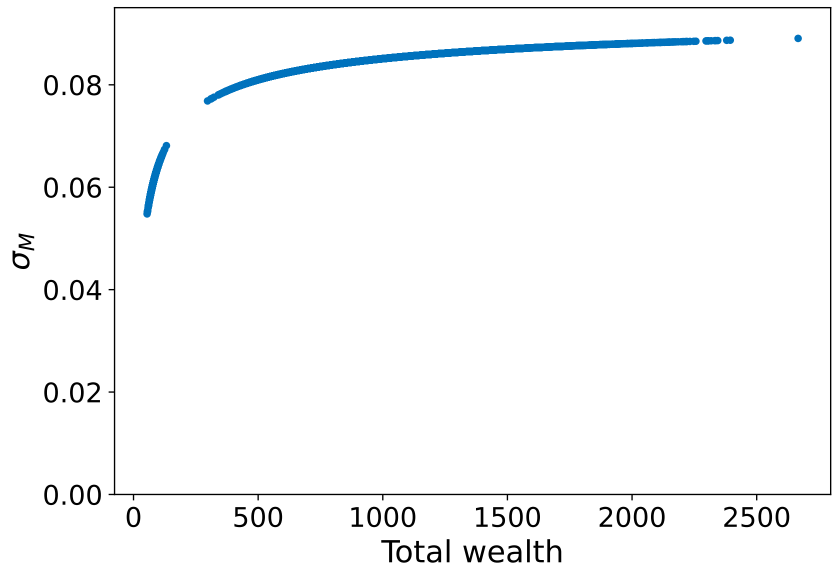


Figure 15: Frisch EIS for non-rentiers under Comin et al. preferences.

Notes: The figure plots the first-period Frisch elasticity of intertemporal substitution, σ_0, for simulated non-rentiers under the Comin et al. preference specification. This is the EIS that enters the sufficient-statistic formula when preferences are not time separable.

F Appendix for Section 6: Illustrative Application to US Data

F.1 Data Construction

We use two different datasets. One is a household-level survey (PSID) and the other is data on prices of different categories of goods (CPI). The PSID is a longitudinal survey, interviewing households annually until 1997 and biennially thereafter. Each sample includes about 7,000-9,000 households. We use seven spending categories and merge them with CPI categories. We describe how we construct the variables needed for our methodology below.

Net Assets:

The wealth module of the PSID tracks the value of components of household balance sheets (business equity, stocks, mutual funds, bonds, automobiles, pensions, cash, etc.). Home equity data are recorded as the value of a household's home minus its mortgage obligations. The PSID aggregates these variables, imputes missing values, and reports the comprehensive variables WEALTH1 and WEALTH2. WEALTH1 represents wealth excluding home equity, while WEALTH2 is the sum of WEALTH1 and home equity. As Cooper et al. (2019) note, these measures exclude the value of defined-contribution (DC) account. We define net assets as WEALTH2 plus the value of DC account (recorded separately in the PSID) to incorporate as much of the household's assets as possible.[36]

Capitalized wealth proxy:

We construct a proxy for total wealth by adding the capitalized value of labor income and transfers to net assets. Define household income as labor income plus the variables recorded as social security income and other welfare income. First, we estimate the age-specific income profile for each period τ using cross-sectional data. To do this, we regress a quadratic of the age of the head of household on log income controlling for household characteristics (marital status, state of residence, race of household head, gender, and occupation) and year fixed effects. We then use this regression to predict each household's income profile as their age increases. We inflate these predictions of the household's income in the future by an estimate of expected nominal per capita GDP growth. The expected growth in nominal GDP comes from the Congressional Budget Office's real-time (contemporaneous) forecast of nominal GDP growth and the population growth rate uses realized population growth rates for the United States, assuming a constant growth after 2019. We discount these nominal income flows back to the present using a nominal rate of 6%, consisting of a 4% real rate, following Catherine et al. (2022), and a 2% expected inflation rate. We assume that income flows are zero beyond age 90.

Owner-occupied housing:

For renters, we use the housing expenditures variable in the PSID (which includes utilities). For owner-occupied housing, we impute housing costs by matching homeowners to renters using static budget shares in each period. This procedure should yield accurate estimates as long as preferences are time separable.

Specifically, for each year, we run the following regression for renters:

$$housing_{h,\tau} = \sum_{i \neq \text{housing}} \alpha_{i,\tau} spending_{i,h,\tau} + \beta_{1,\tau} age_{h,\tau} + \beta_{2,\tau} age^2_{h,\tau} + stateFE_{h,\tau} + \epsilon_{h,\tau},$$

where the left-hand side variable is expenditures on housing (including utilities), and co-

[36]Cooper et al. (2019) report that adding DC account information to WEALTH2 generally matches the total assets reported in the Survey of Consumer Finances (SCF). If no value was provided and the value was given in bins, the median household value between the bins was used for imputation.

variates are households' spending on non-housing categories, age, and state fixed effects. We then use this regression to impute (predict) rental expenditures for homeowners based on their age, spending on non-housing categories, and state of residence.

In 2019, a new question was added to the PSID survey which asks the following:

> *If someone were to rent this (apartment/mobile home/home) today, how much do you think it would rent for per month, unfurnished and without utilities?*

We use the responses to this question to validate our procedure. A regression of the survey values (including utilities) on our imputed values, both relative to current expenditures, has a coefficient of 1.03 with an R^2 value of 0.59. This suggests that our imputation performs well.

Budget shares:

We align the seven categories of the PSID (food, housing, transportation, education, health, clothing, and recreation) with the CPI.[37] As mentioned above, for homeowners, we impute housing costs. The relative budget share is defined as the spending on each category divided by total spending. We compute the consumption-wealth ratio of households by dividing total spending in each year by wealth.

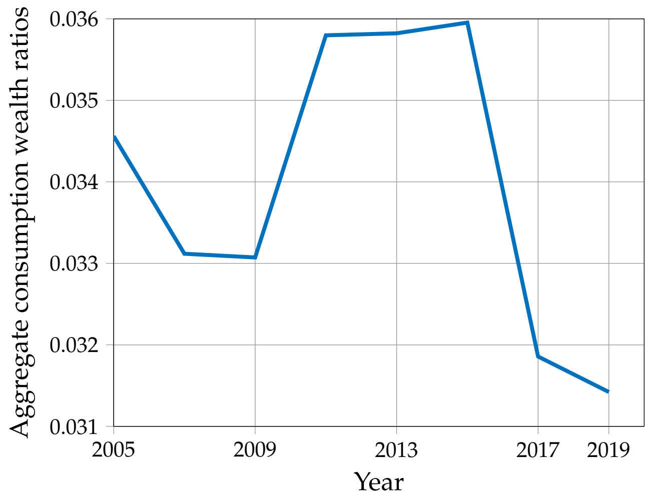
F.2 Additional Figures

Figure 16 plots total consumption expenditures relative to proxy wealth in our data. On average, households spend around 3% of their total proxy wealth in the present. The aggregate consumption to wealth ratio rose during the great recession, as wealth shrank relative to consumption, but fell during the stock market boom in the mid 2010s. Over the whole sample, the aggregate consumption-wealth ratio fell.

Figure 17 displays the distribution of money-metric wealth, in 2019, for all age groups, including rentiers and non-rentiers. Figure 18 plots nominal log money-metric wealth against nominal capitalized (proxy) wealth for rentiers and non-rentiers. If financial markets were complete and absent measurement error, these two figures should be 45 degree lines.

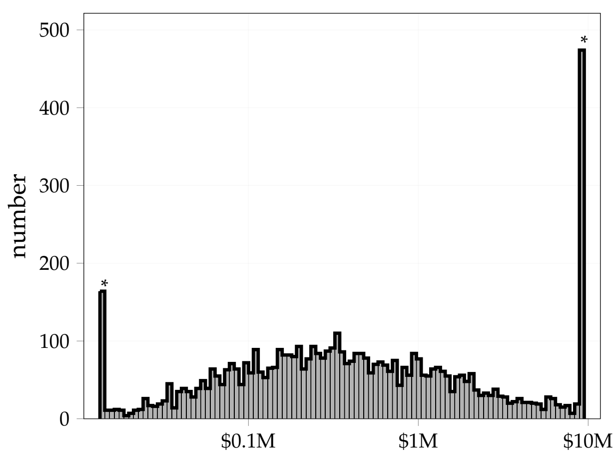
[37]The corresponding codes for CPI are CPIFABSL, CPIHOSSL, CPITRNSL, CPIEDSL, CPIMEDSL, CPIAPPSL, and CPIRECSL, respectively. Education includes child care. Recreation includes Trips & vacations and Recreation & entertainment in PSID.

Figure 16: Aggregate consumption-to-wealth ratio over time



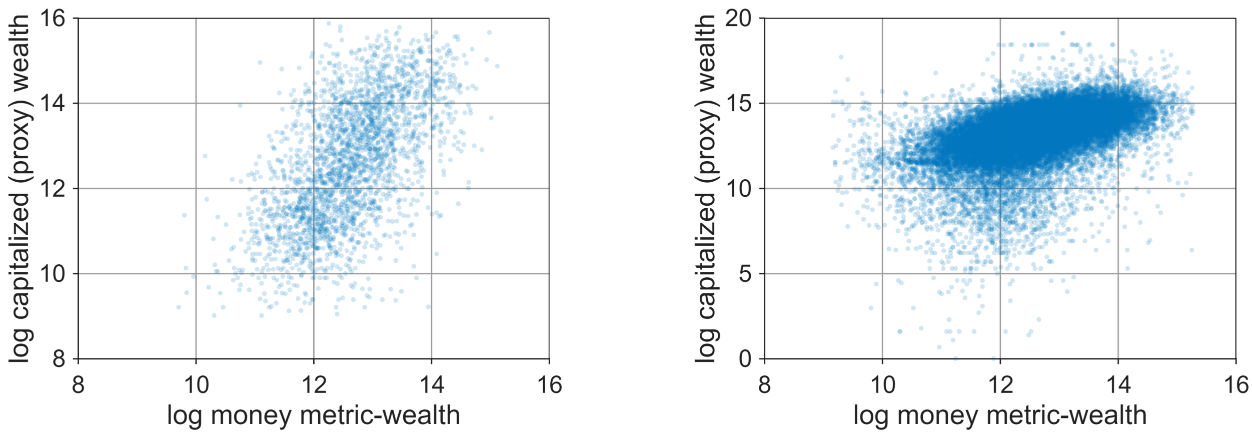
Notes: Aggregate consumption is the sum of consumption expenditures for all households in our sample. Aggregate wealth is the sum of proxy wealth for all households in our sample.

Figure 17: Distribution of money-metric wealth in 2019



Notes: Bars with asterisk are households whose estimated money-metric wealth in $2019 dollars is outside of the wealth range of rentiers in that year.

Figure 18: Capitalized (proxy) wealth against money-metric wealth (in logs) for all years



(a) Rentiers (b) Non-rentiers

Notes: All variables in this figure are in nominal terms (not base year prices).